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Magnetic Helicity and the Calabi Invariant

HELICITY 2020

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Motivation

- Classical helicity measures linkage of magnetic flux

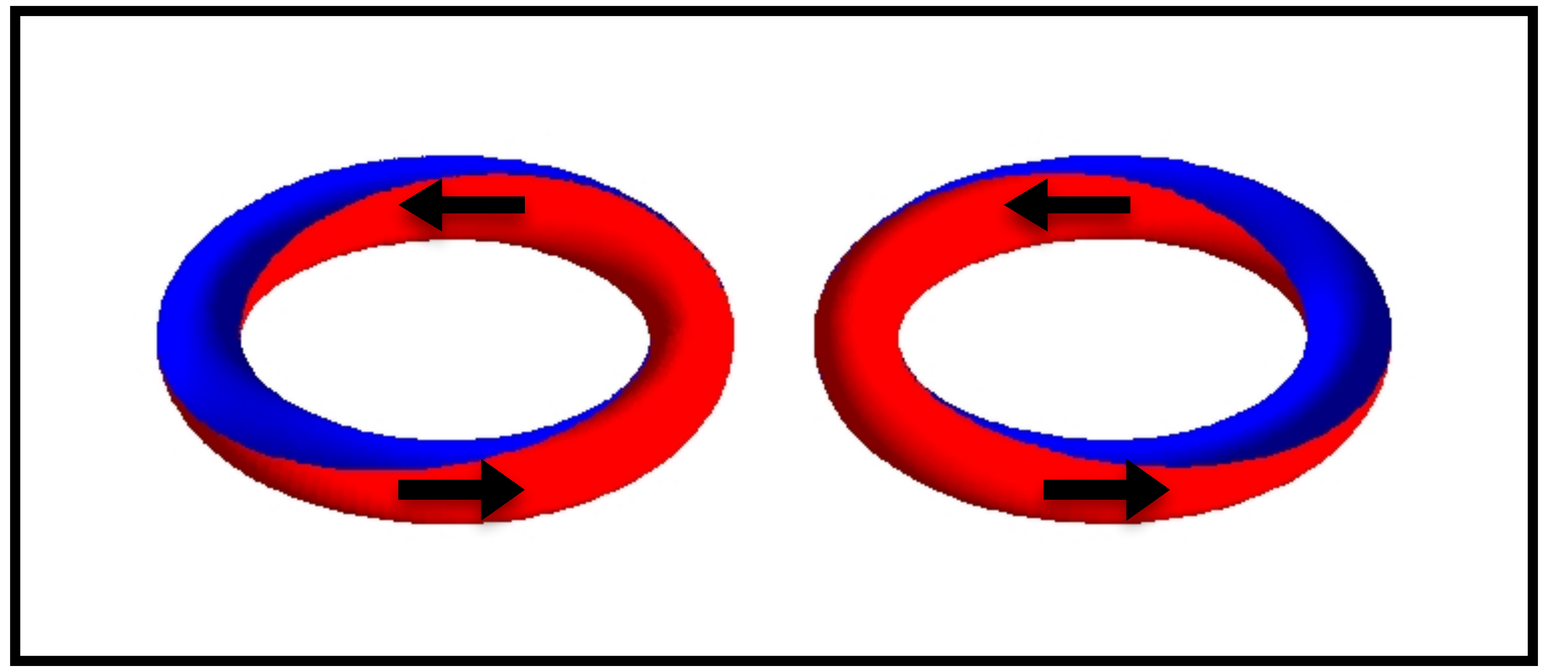
$$H(\mathbf{B}) = \int_V \mathbf{A} \cdot \mathbf{B} \, dV, \quad \nabla \times \mathbf{A} = \mathbf{B}, \quad \mathbf{n} \cdot \mathbf{B}|_{\partial V} = 0.$$

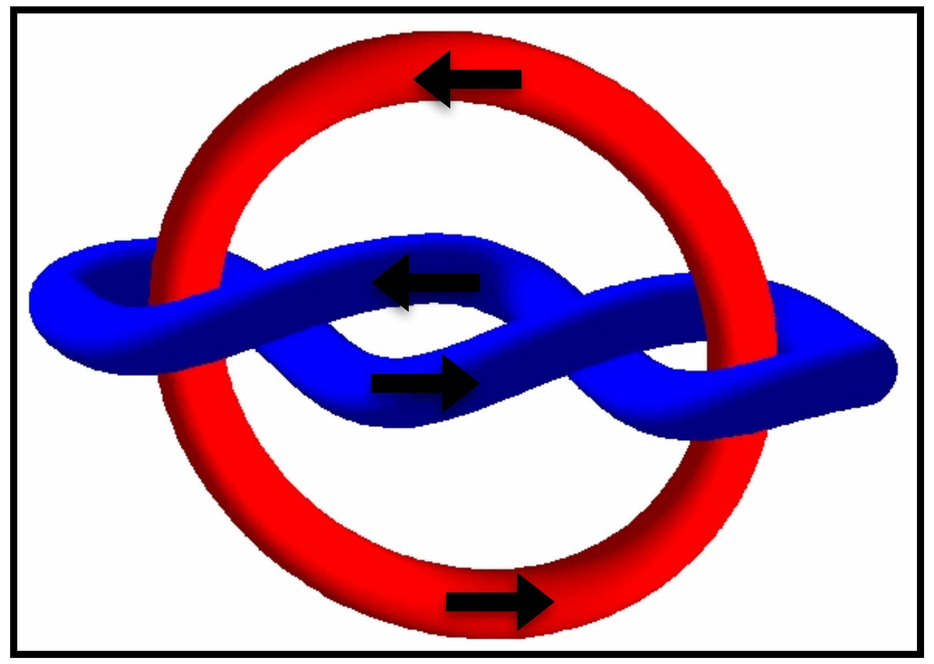
- However, domains that are bounded by a flux surface are rare. In most cases we have to use the relative helicity (Berger & Field 1984), (Finn & Antonsen 1985)
  $H_R = \int_V (A + A_{\mathrm{ref}}) \cdot (\mathbf{B} - \mathbf{B}_{\mathrm{ref}}) \; d^3x$

- Additional complication: expression depends on the choice of reference field. Interpretation becomes difficult when the reference field is time-dependent.

Motivation

- Helicity is an integral over the domain, it does not detect any substructures with net zero helicity





- Attempts to overcome extract more detailed information from the linking structure have led to the development of field line helicity.

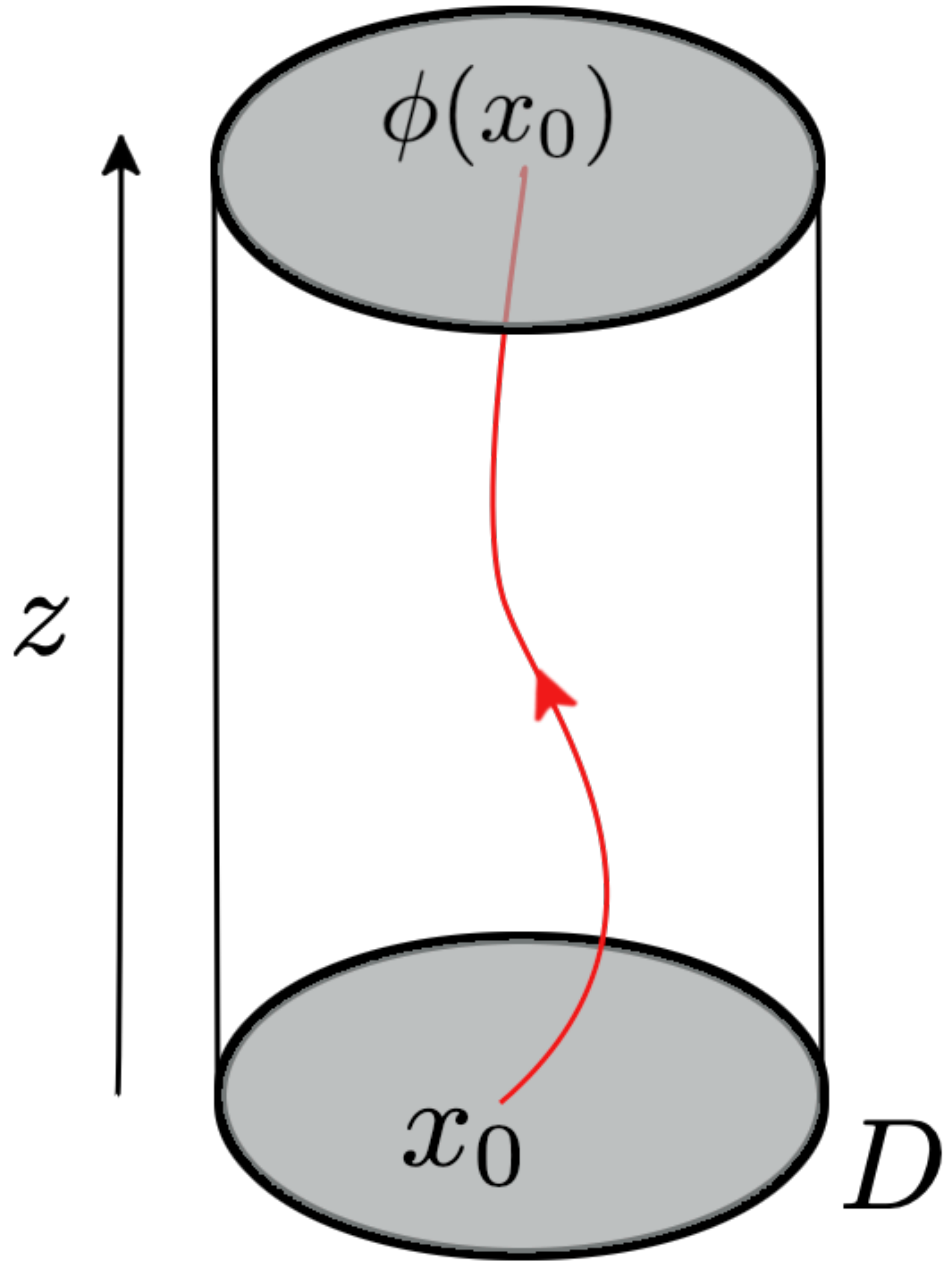
Field Line Helicity

Field line mapping generated by

$$\frac{d}{dz}\phi(x_0, z) = \frac{\mathbf{B}(\phi(x_0,z))}{B_z(\phi(x_0,z))}$$

Field line helicity is defined as

$$\mathcal{A}(x_0) = \int_{x_0}^{\phi(x_0)} \mathbf{A} \cdot \mathrm{dl}$$
$$= \int_0^1 \mathbf{A}(\phi(x_0), z) \cdot \frac{\mathbf{B}(\phi(x_0),z)}{B_z(\phi(x_0),z)}\, dz$$



Field Line Helicity and Relative Helicity

Choose a reference field with vector potential

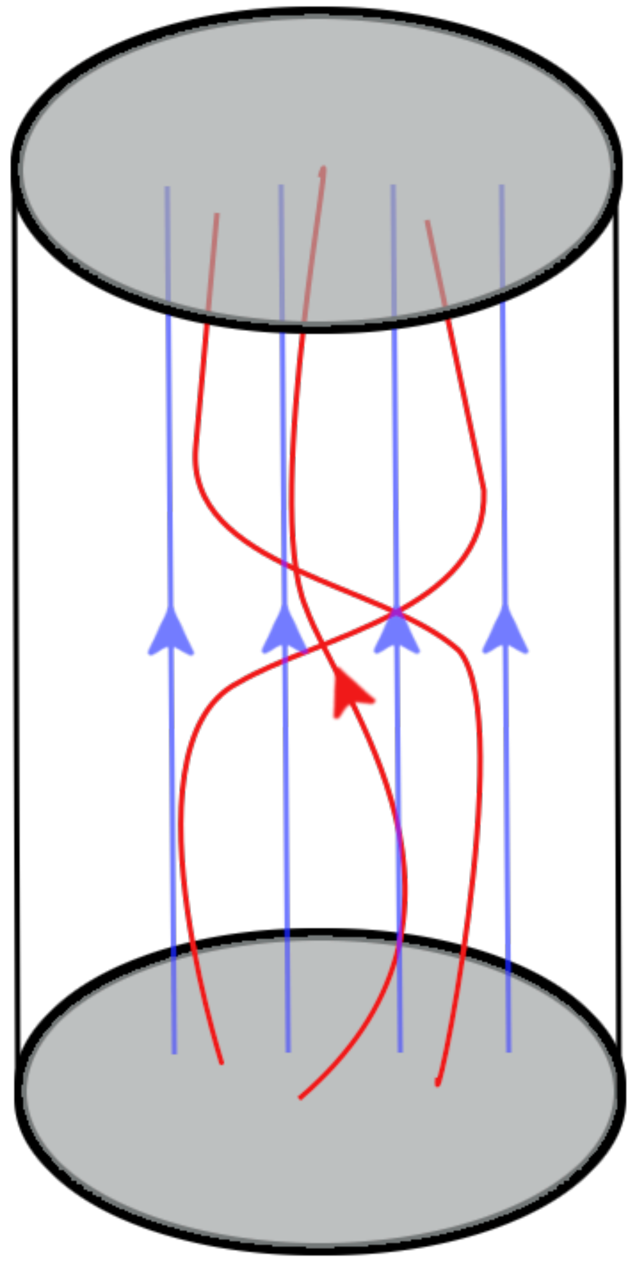
$$\nabla \times \mathbf{A}_{\mathrm{ref}} = \mathbf{B}_{\mathrm{ref}}$$

Relative helicity is given by

$$H(\mathbf{B};\mathbf{B}_{\mathrm{ref}}) = \int_V (\mathbf{A} + \mathbf{A}_{\mathrm{ref}}) \cdot (\mathbf{B} - \mathbf{B}_{\mathrm{ref}})\; d^3x$$

Fix the gauge as $\mathbf{A} \times \vec{n}|_{\partial V} = \mathbf{A}_{\mathrm{ref}} \times \vec{n}|_{\partial V}$, then

$$H(\mathbf{B};\mathbf{B}_{\mathrm{ref}}) = \int_D B_z(x_0)\mathcal{A}(x_0)\; d^2x_0$$



Properties of Field Line Helicity

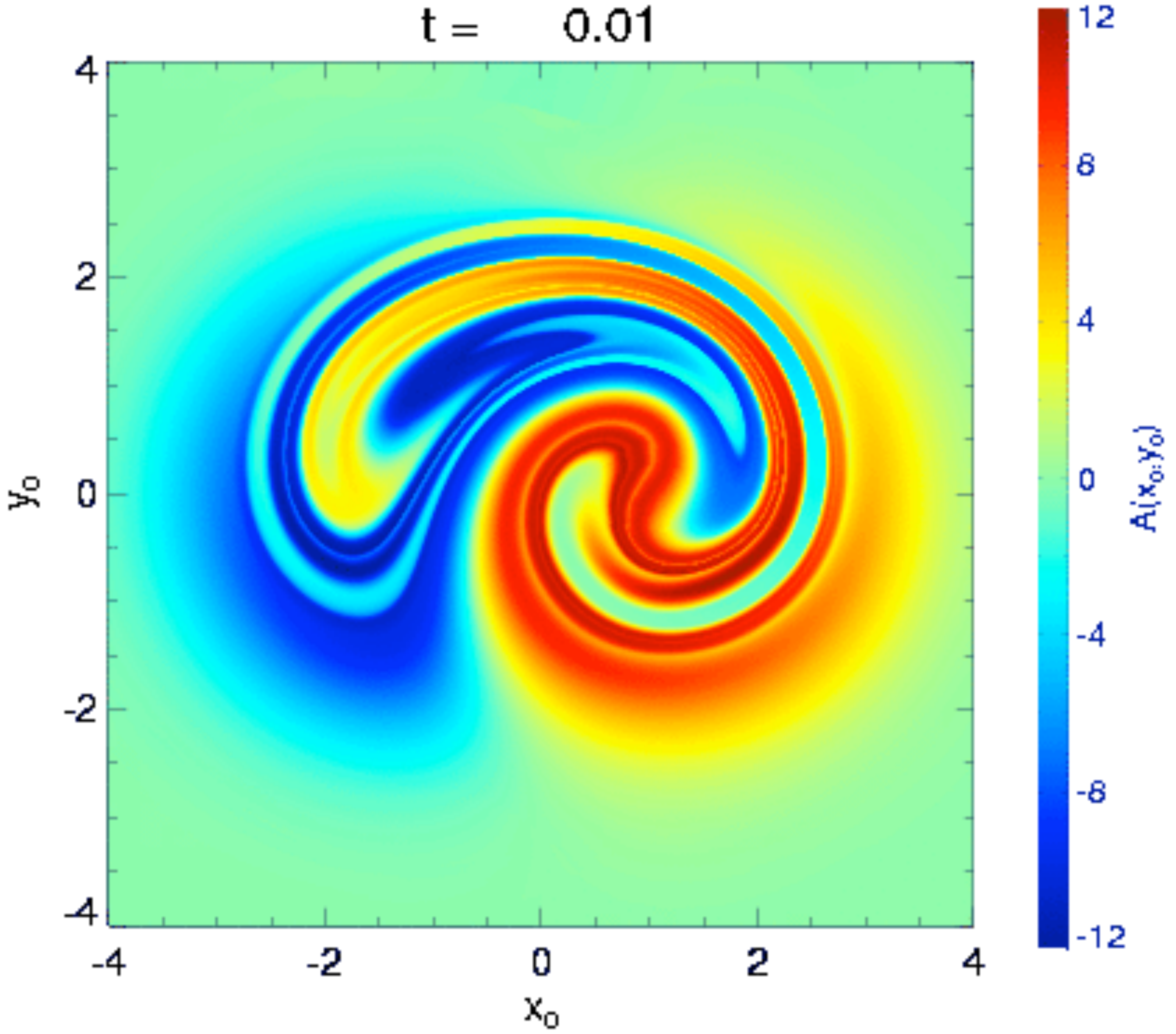
$$\mathcal{A}(x_0) = \int_{x_0}^{\phi(x_0)} \mathbf{A} \cdot \mathrm{dl} \qquad\qquad \mathbf{A} \times \vec{n}|_{\partial V} = \mathbf{A}_{\mathrm{ref}} \times \vec{n}|_{\partial V}$$

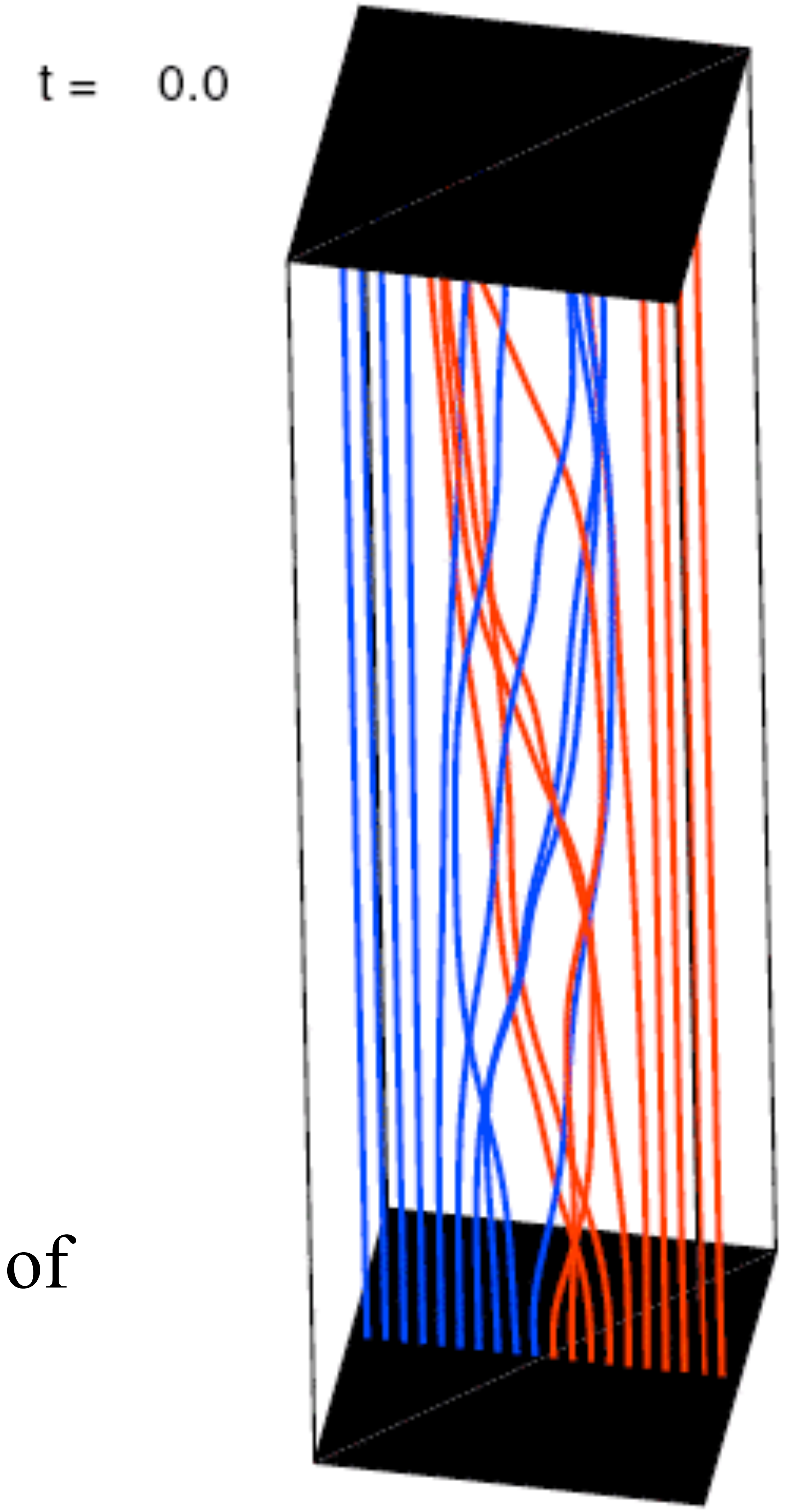
- Under the above gauge condition, the field line helicity is gauge invariant only on periodic field lines.

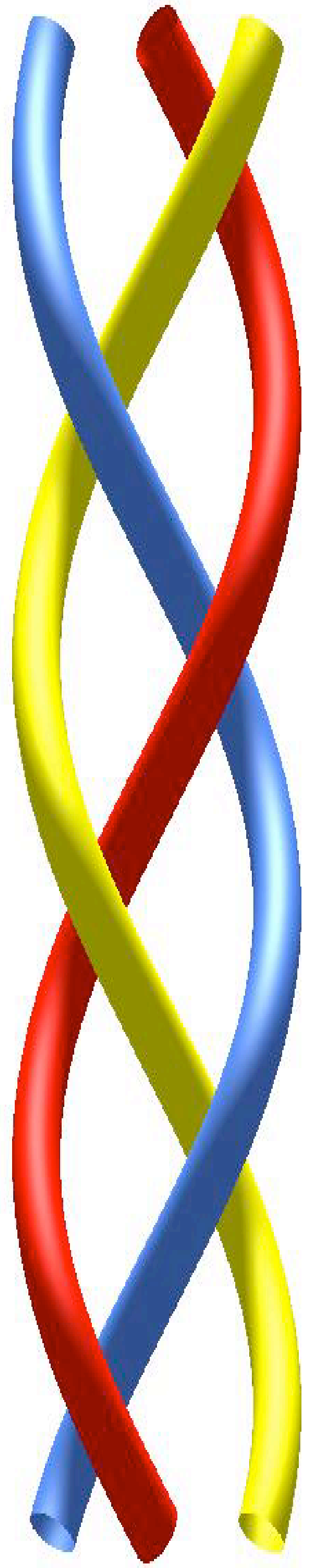
$$\mathbf{A}_{\mathrm{ref}} \quad \rightarrow \quad \mathbf{A}_{\mathrm{ref}} + \nabla\Psi \qquad \text{implies} \qquad \mathcal{A}(x_0) \quad \rightarrow \quad \mathcal{A}(x_0) + \Psi(\Phi(x_0)) - \Psi(x_0)$$

- It is, however, an invariant under ideal deformations of the flux tube domain that leave the boundary fixed.
- It also *uniquely determines* the topology of the magnetic field. [Yeates & Hornig Physics of Plasmas 20.1 (2013)]
- Physically the field line helicity can be interpreted, for a single field line, as the averaged flux between the field line and the side boundary of the flux tube.

Field Line Helicity: Example







H = 0 but the evolution of $A(x_0)$ reveals the presence of additional constraints. E.g. there is no annihilation between positive and negative regions of $A(x_0)$.
[Yeates, Hornig & Wilmot-Smith PRL 105 (2010)]

The Calabi Invariant

The Calabi invariant is a real-valued map on the space of smooth functions on a disk D, that preserves the symplectic 2-form or area form.

$$\mathrm{Cal}: C^{\infty}(D, \omega_{\mathrm{area}}, \partial D) \to \mathbb{R}$$

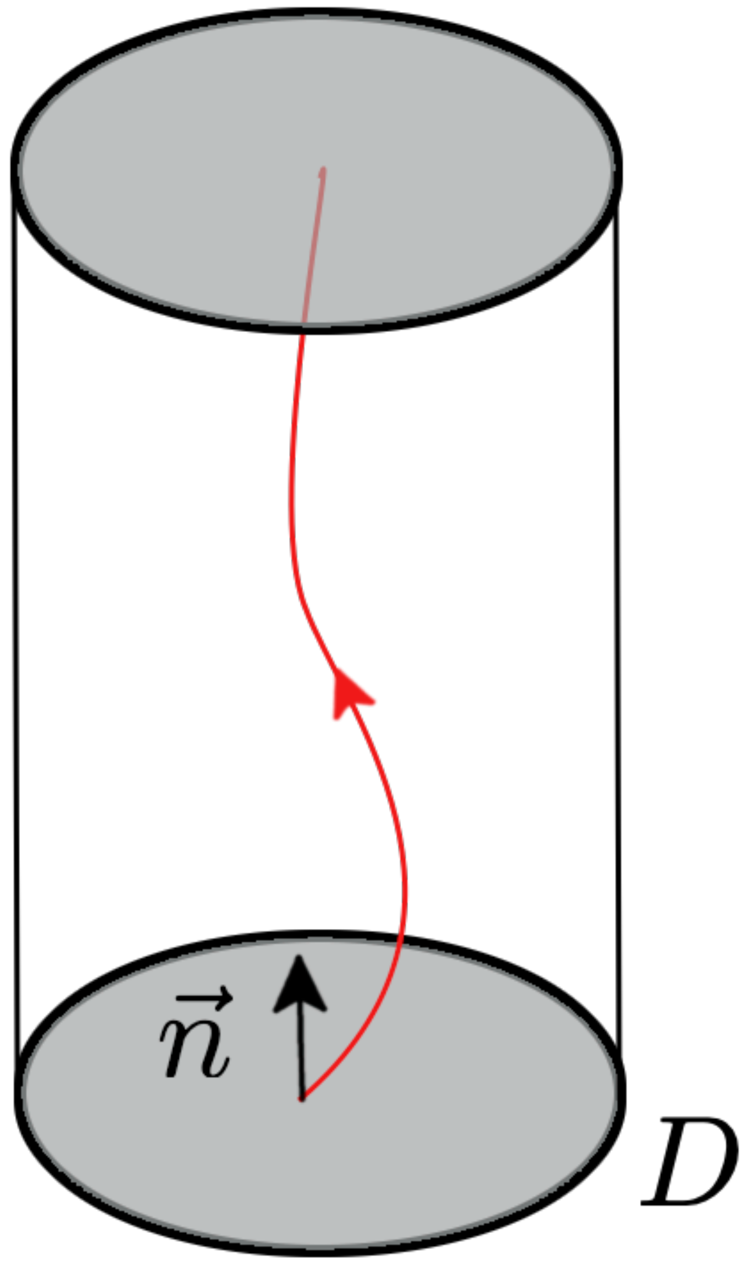
It can be shown that the field line mapping belongs to the space of smooth functions,

$$\phi \in C^{\infty}(D, \omega_{\mathrm{area}})$$

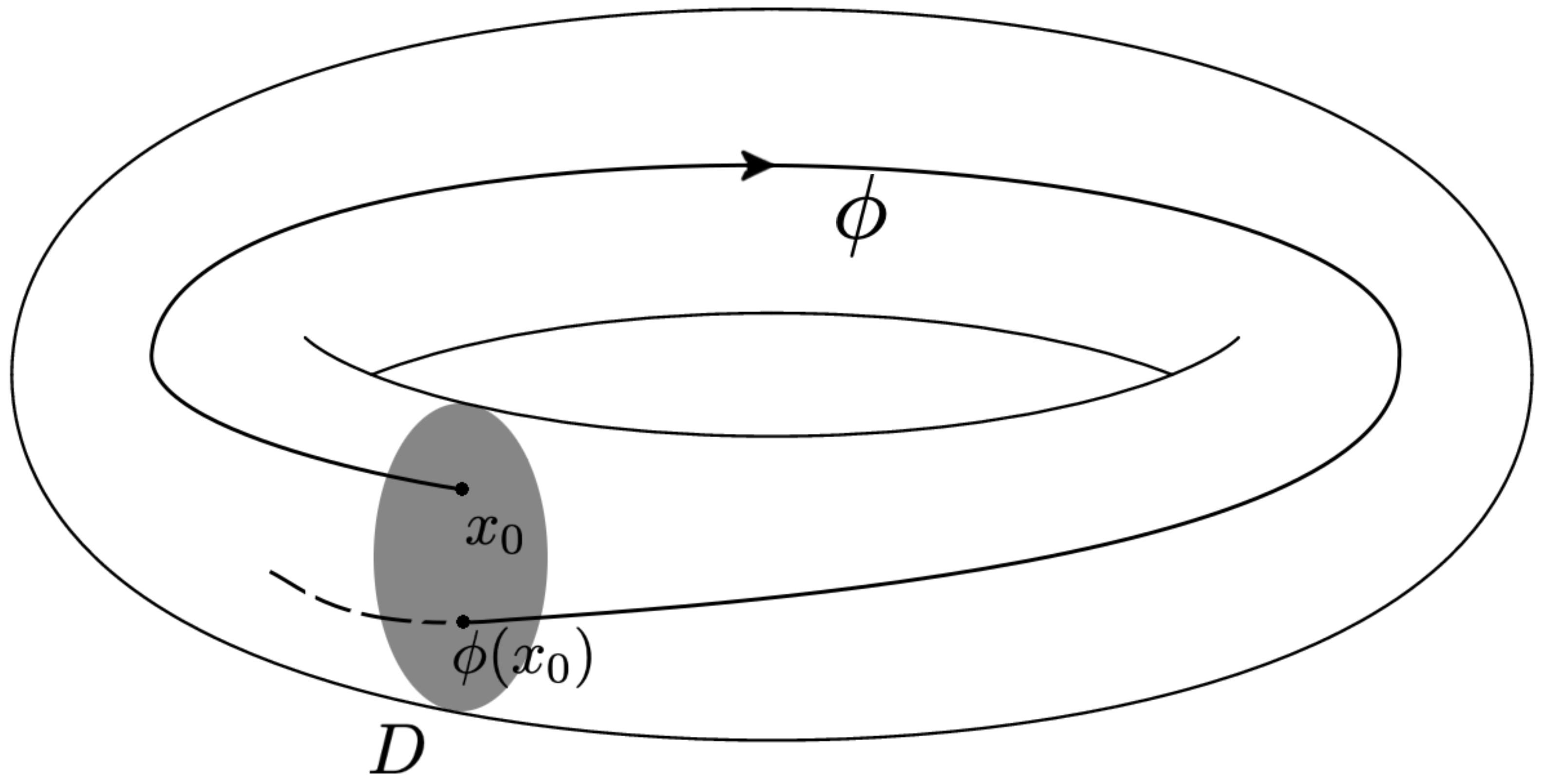
where a subset of field line mappings also preserve the boundary.

The area 2-form is defined, in the flux tube geometry, as

$$\omega_{area} = \mathbf{B} \cdot \mathbf{n}\, dx_0^2$$



The Calabi Invariant



Define the Calabi invariant (Calabi 1970) as

$$\mathrm{Cal}(\phi) := \int_D h(x)\, \omega_{\mathbf{area}}$$

For a cylindrical domain this becomes

$$\mathrm{Cal}(\phi) = \int_D h(x)\ B_z(x)\ d^2x$$

where h is a smooth function satisfying $dh = \phi^*\alpha - \alpha\,,\ \ h|_{\partial D} = 0\,,\ \Phi|_{\partial D} = Id$, and $d\alpha = \omega$

The above can be reformulated to obtain

$$\mathrm{Cal}(\phi) = -\int_D \phi^*\alpha \wedge \alpha$$

Comparison with (field line) helicity shows : $\mathrm{Cal}(\phi) = H(\mathbf{B})$ and $\mathcal{A} = h$

The Calabi Invariant

Differential Geometry	Vs. Vector Analysis
$Cal(\Phi) = \int_D h(x,y)\omega_{area}$	$Cal(\Phi) = \int_D h(x,y) B_z(x,y)\, dx\, dy$
$d\alpha = \omega_{area}$	$\nabla \times \vec{a} = B_z(x,y)\vec{e}_z$ $\vec{a}(x,y) = a_x \vec{e}_x + a_y \vec{e}_y = -\frac{1}{2}\int_{y_c}^{y} B_z(x,y')dy'\vec{e}_x + \frac{1}{2}\int_{x_c}^{x} B_z(x',y)dx'\vec{e}_y$
$dh = \phi^* \alpha - \alpha$	$\nabla h(x,y) = \left(a_x(\Phi^x,\Phi^y)\frac{\partial \Phi^x}{\partial x} + a_y(\Phi^x,\Phi^y)\frac{\partial \Phi^y}{\partial x} - a_x(x,y) \right)\vec{e}_x + \left(a_x(\Phi^x,\Phi^y)\frac{\partial \Phi^x}{\partial y} + a_y(\Phi^x,\Phi^y)\frac{\partial \Phi^y}{\partial y} - a_y(x,y) \right)\vec{e}_y$

A Boundary Correction Term

For case where the field line mapping is not identity on the boundary, we can separate the field line mapping into two component flows

$$\phi = \tilde{\phi} \circ \phi_\varphi$$

where $\tilde{\phi}$ indicates the flow from footpoints on the interior of the disk and ϕ_φ is the flow restricted to a rotation on the boundary.

We can then calculate the Calabi invariant as follows:

$$\mathrm{Cal}(\phi) = \mathrm{Cal}(\tilde{\phi} \circ \phi_\varphi) = \mathrm{Cal}(\tilde{\phi}) + \mathrm{Cal}(\phi_\varphi)$$

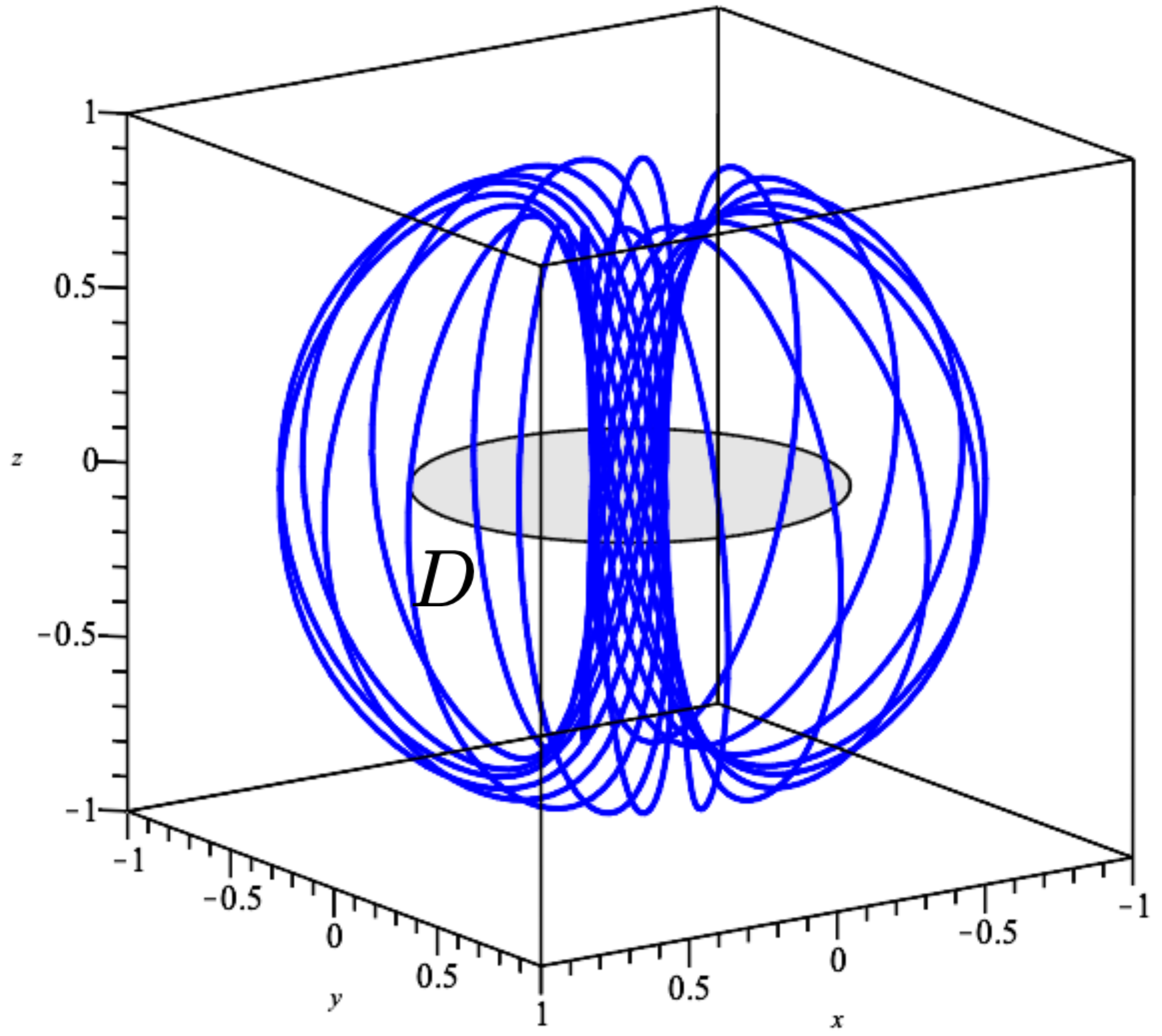
$$\mathrm{Cal}(\phi_\varphi) = 2\pi\theta\alpha_\varphi^2(R)$$

In the radially symmetric case we can interpret the turning angle θ as

$$\theta = \frac{1}{2\pi}\int_0^{2\pi} \phi_\varphi(R)\, d\varphi$$

Calculating with the Calabi Invariant

Example: Consider a spherical force-free field.



$$\mathbf{B} = u(r,\theta)\ \vec{e_r} + v(r,\theta)\ \vec{e_\theta} + w(r,\theta)\ \vec{e_\varphi}$$

$$u(r,\theta) = r^{-3/2} J_{\frac{3}{2}}(\lambda r)\cos(\theta),$$

$$v(r,\theta) = -\frac{1}{2r}\frac{\partial}{\partial r}(r^{1/2} J_{\frac{3}{2}}(\lambda r))\sin(\theta),$$

$$w(r,\theta) = \frac{\lambda}{2r^{1/2}} J_{\frac{3}{2}}(\lambda r)\sin(\theta)$$

Calculating with the Calabi Invariant

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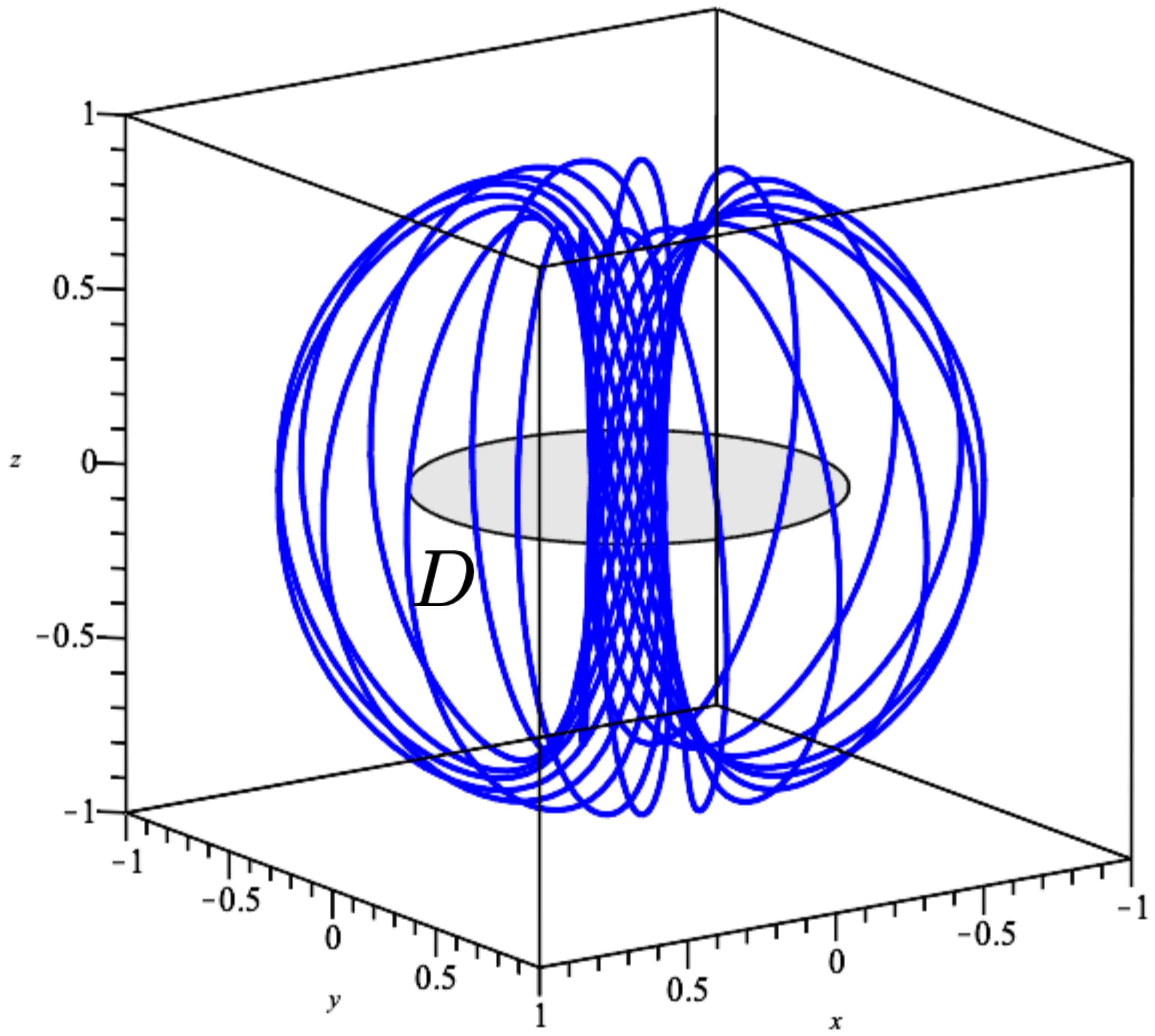

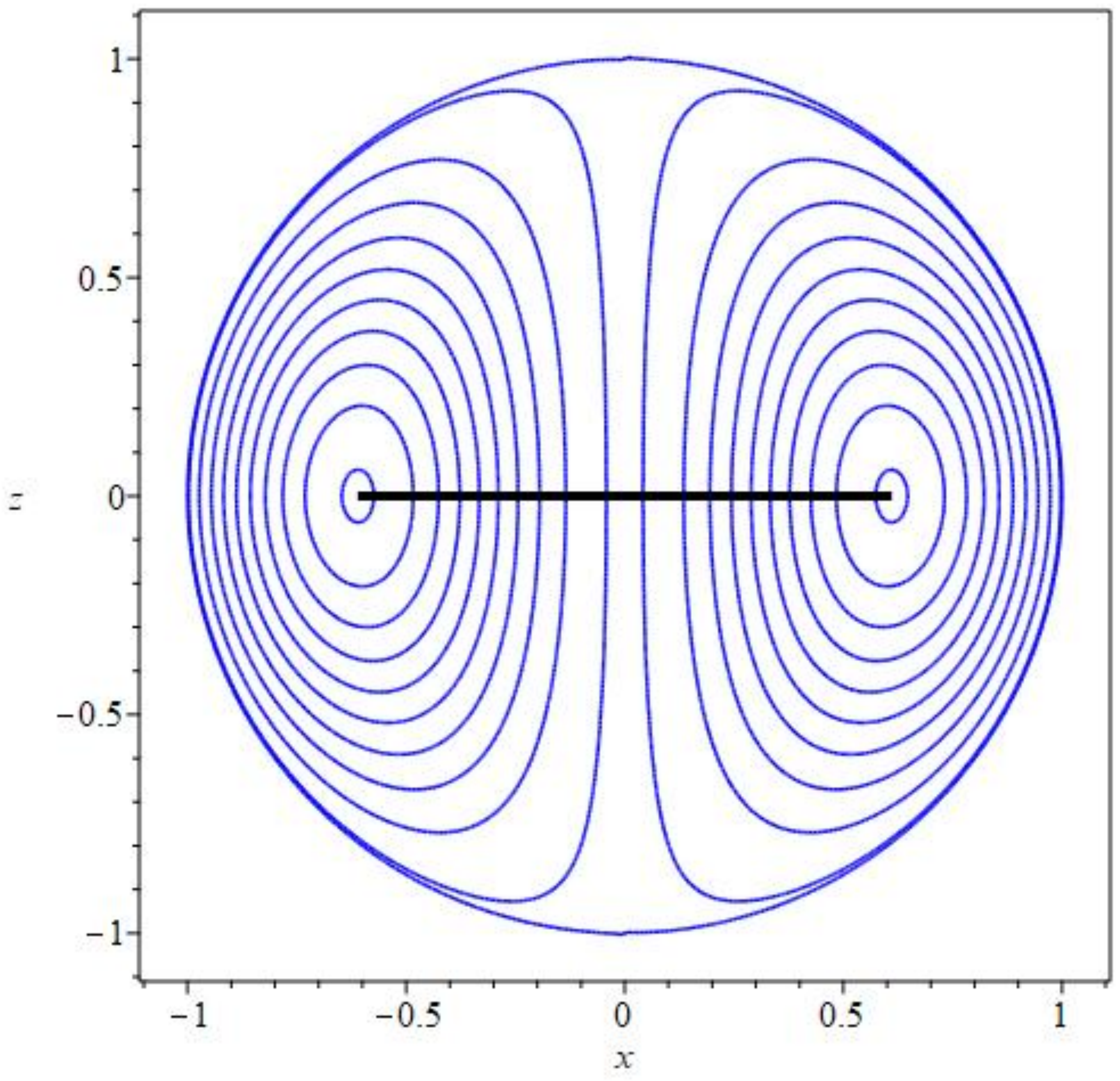

Calculating with the Calabi Invariant

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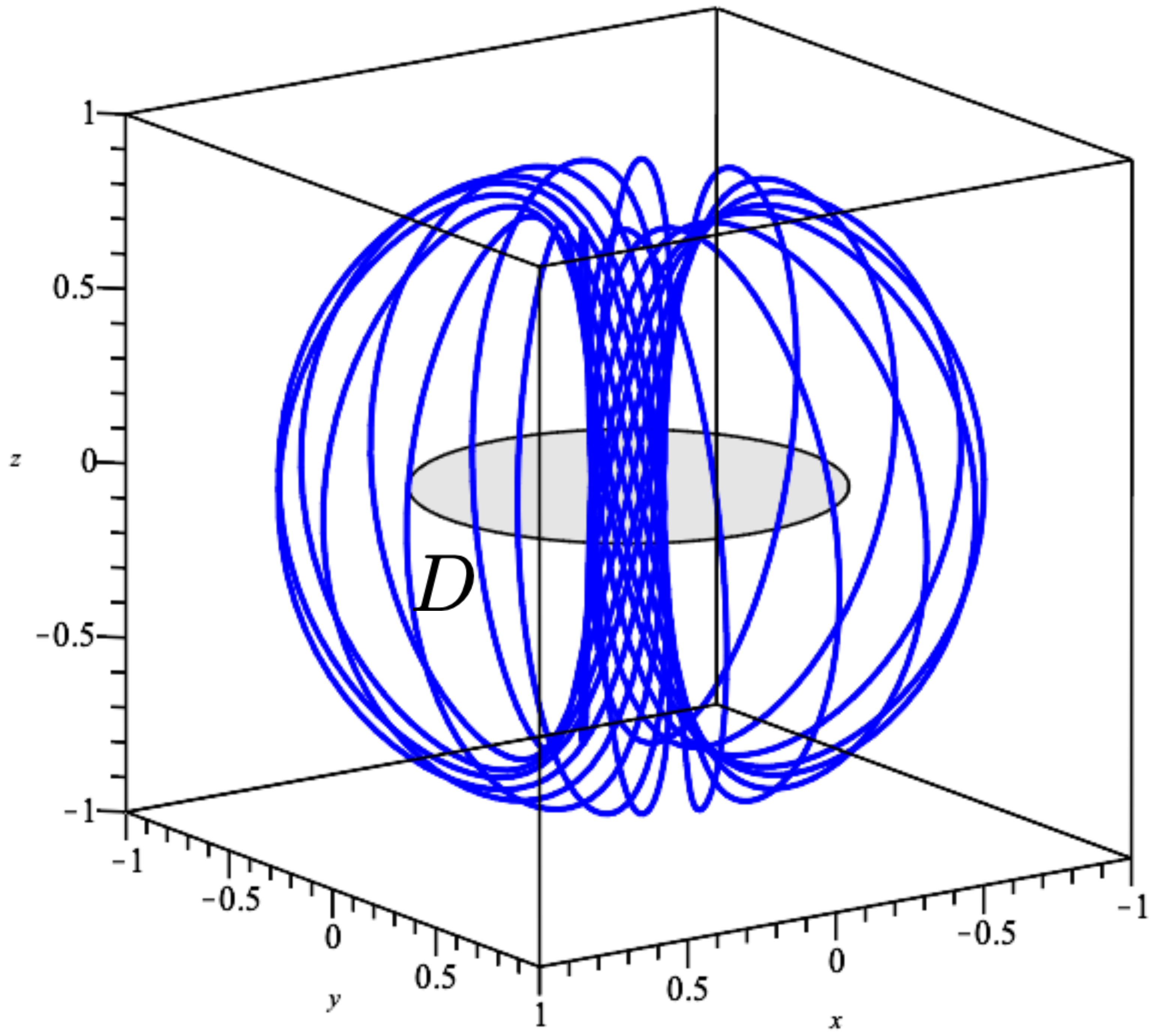
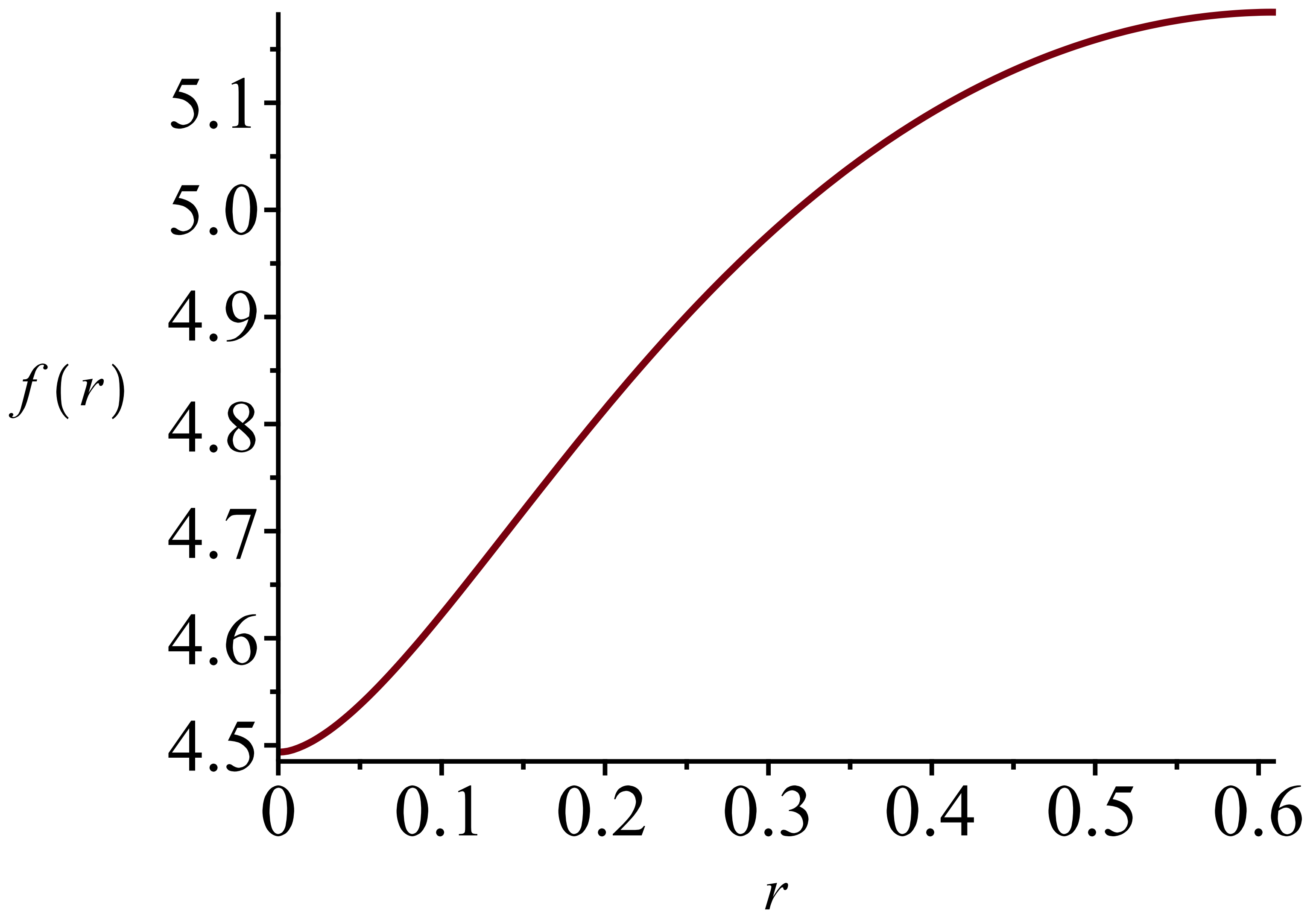


$$\mathrm{Cal}(\phi|_D) = 1.270412733$$

$$H(\mathbf{B}) = 1.270412734$$

Restrict the field line mapping to a cross-sectional disk D,

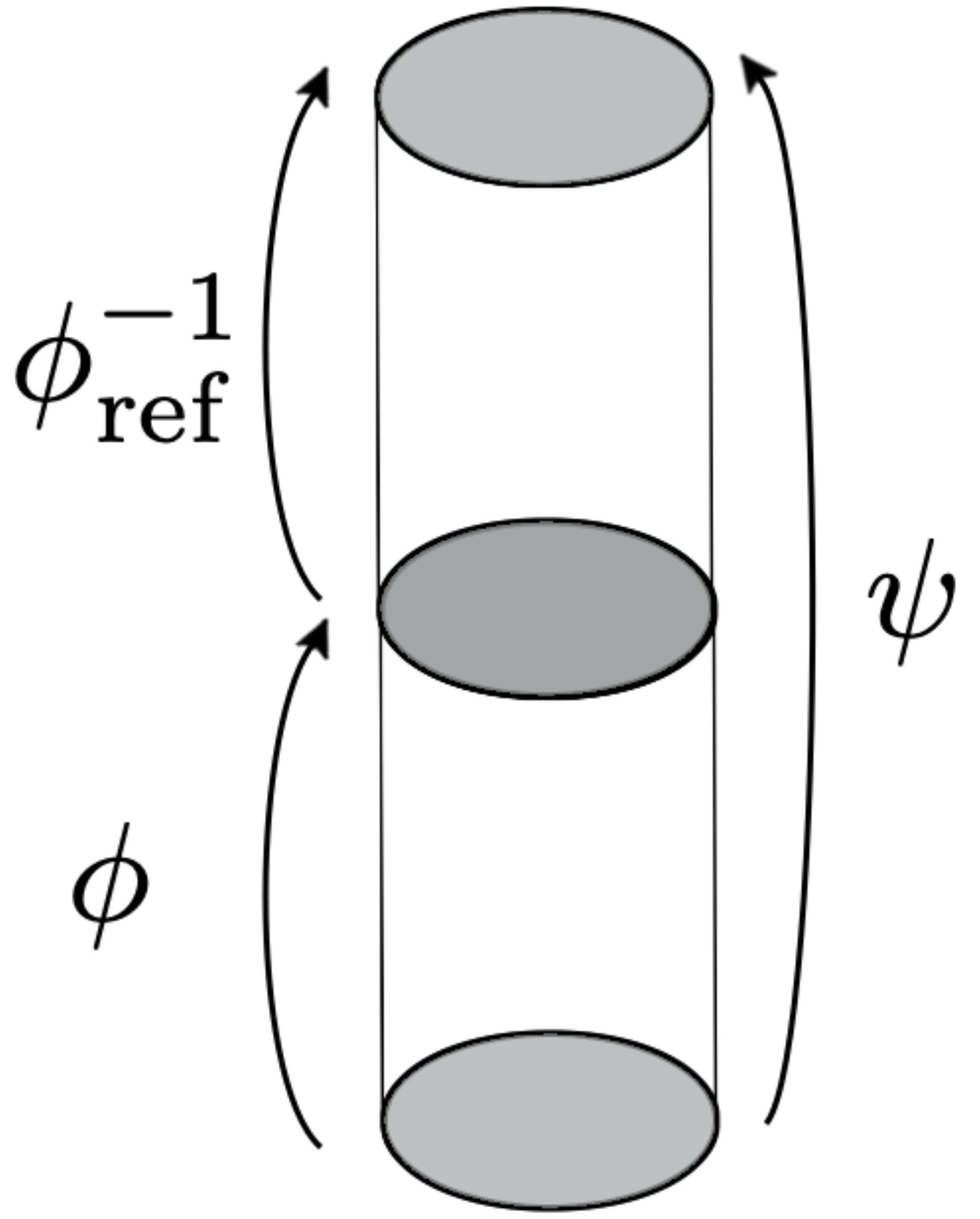
$$\phi|_D : (r, \varphi) \to (r, \varphi + f(r))$$



Relative Helicity and the Calabi Invariant

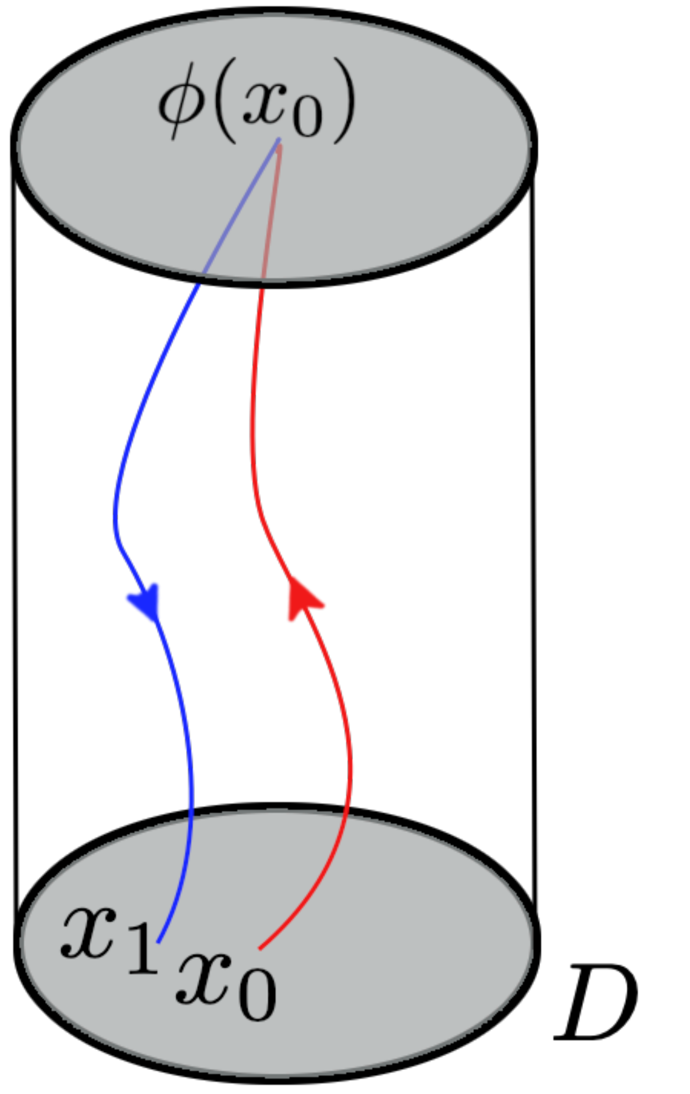
Consider the map $\phi_{\text{ref}}^{-1} \circ \phi : D \to D$ forming an automorphism of the disk.

$$\psi := \phi_{\text{ref}}^{-1} \circ \phi \qquad \frac{d\phi}{ds} = \mathbf{B}(\phi) \qquad \frac{d\phi_{\text{ref}}^{-1}}{ds} = \tilde{\mathbf{B}}_{\text{ref}}(\phi_{\text{ref}}^{-1})$$



The Calabi invariant for this configuration is given by

$$\text{Cal}(\psi) = \int_{V_1} \mathbf{A} \cdot \mathbf{B}\; d^3x + \int_{V_2} \tilde{\mathbf{A}}_{\text{ref}} \cdot \tilde{\mathbf{B}}_{\text{ref}}\; d^3x$$



Relative Helicity and the Calabi Invariant

Define the 'mirror image' under $z \to -z$

$$\frac{d\phi_{\text{ref}}}{ds} = \mathbf{B}_{\text{ref}}(\phi_{\text{ref}})$$

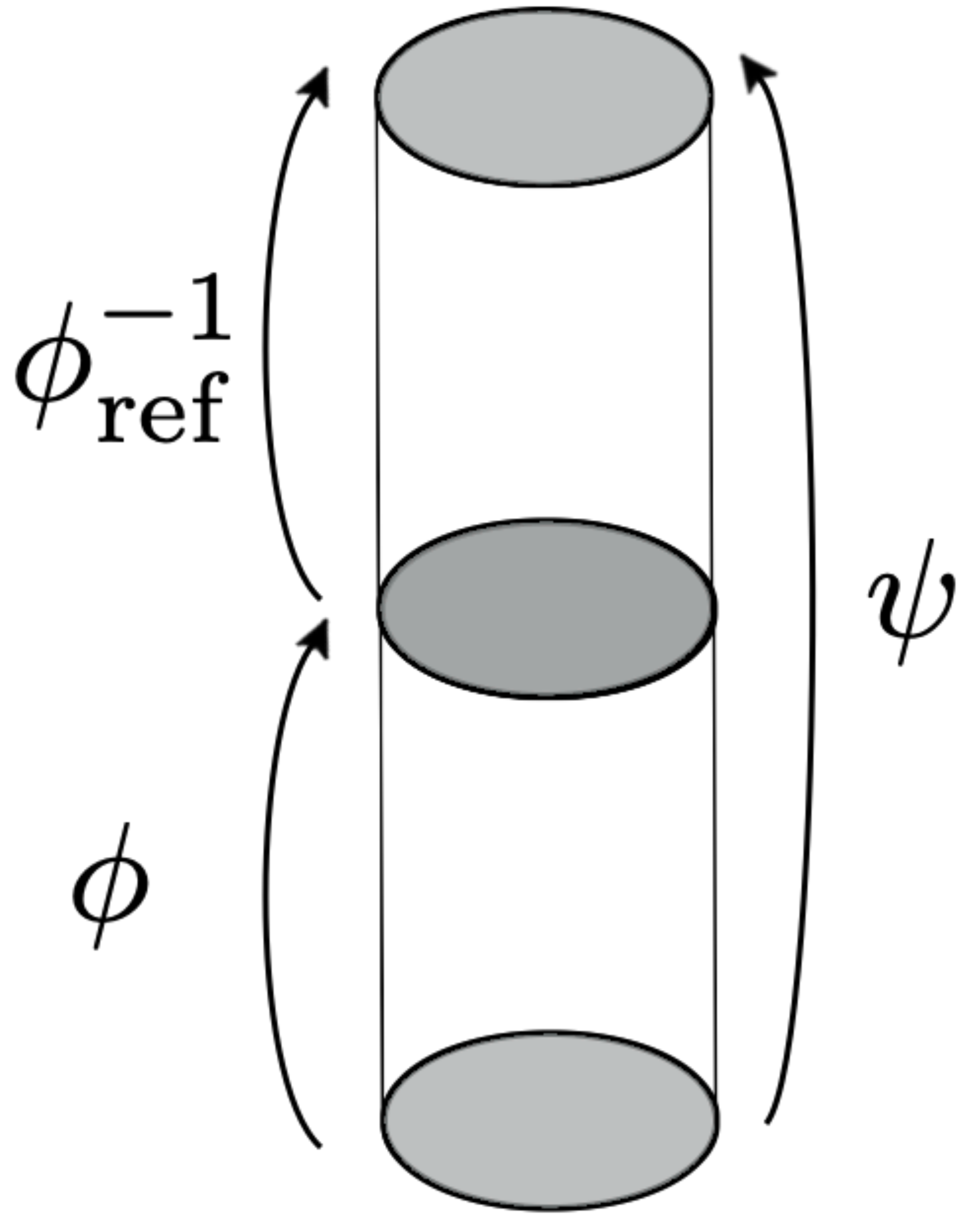
such that

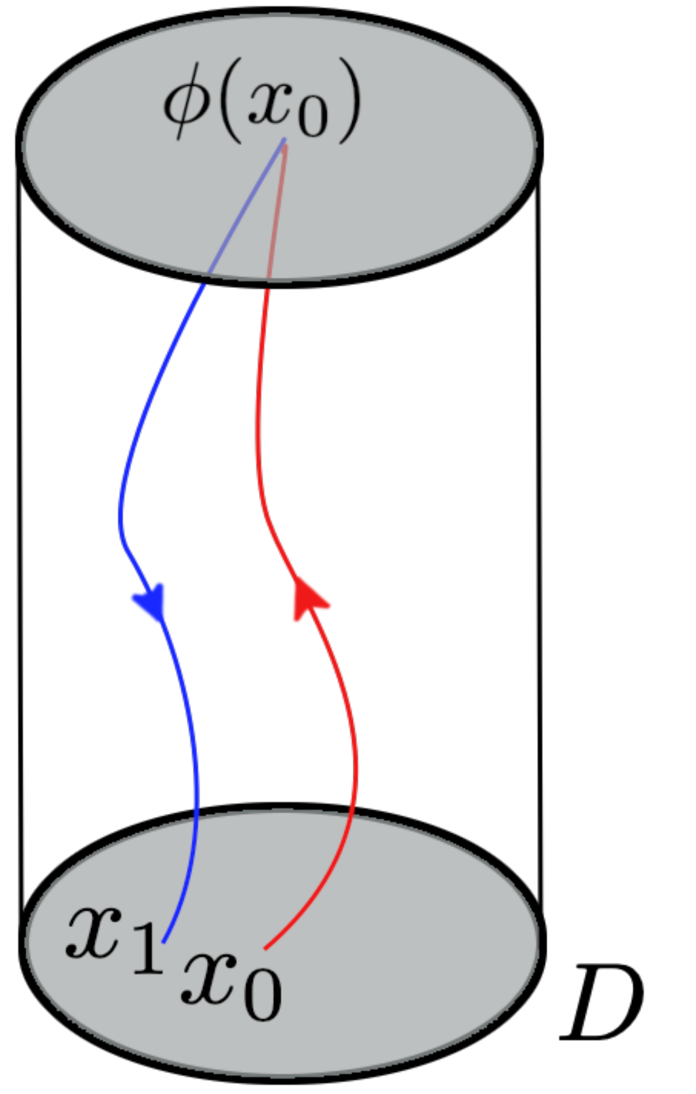
$$\int_{V_2} \tilde{\mathbf{A}}_{\text{ref}} \cdot \tilde{\mathbf{B}}_{\text{ref}} \, d^3x = -\int_{V_2} \mathbf{A}_{\text{ref}} \cdot \mathbf{B}_{\text{ref}} \, d^3x$$

Then the Calabi invariant becomes

$$\text{Cal}(\psi) = \int_{V_1} \mathbf{A} \cdot \mathbf{B} \, d^3x - \int_{V_2} \mathbf{A}_{\text{ref}} \cdot \mathbf{B}_{\text{ref}} \, d^3x$$

$$= H_R(\mathbf{B}, \mathbf{B}_{\text{ref}})$$



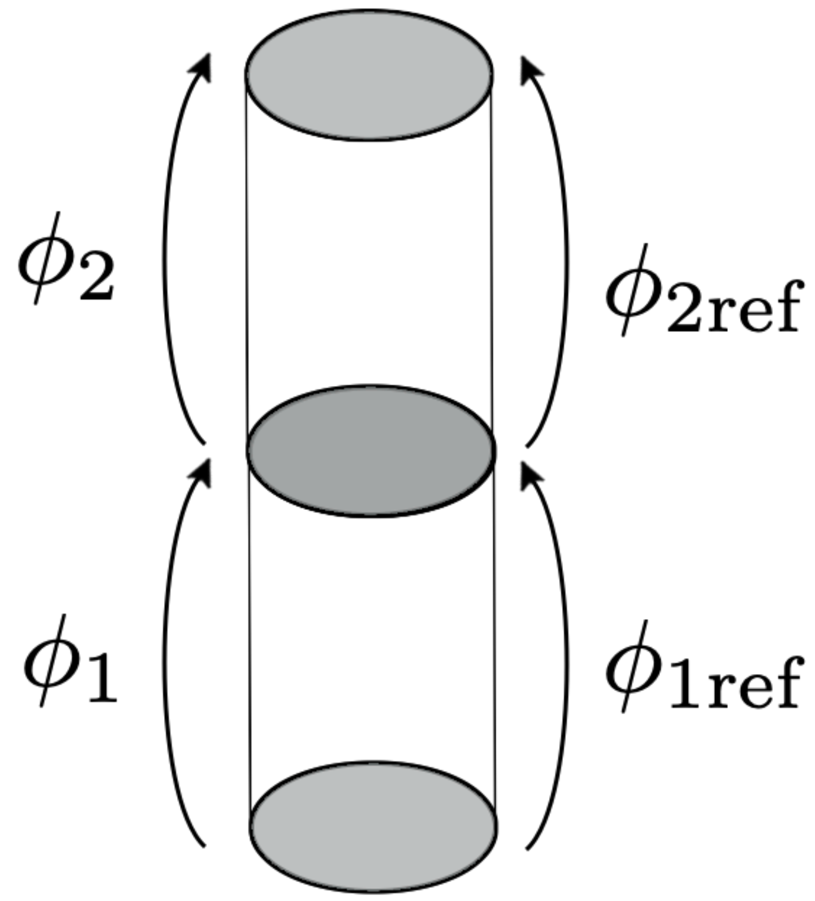


Relative Helicity and the Calabi Invariant

Let V be a volume separated into two sub-volumes V_1 and V_2 along an interface I .

Define magnetic fields by

$$\mathbf{B}(x_0) = \begin{cases} \mathbf{B}_1(x_0) & \text{if } x \in V_1, \\ \mathbf{B}_2(x_0) & \text{if } x \in V_2 \end{cases} \qquad \mathbf{B}_{\text{ref}}(x_0) = \begin{cases} \mathbf{B}_{1\text{ref}}(x_0) & \text{if } x \in V_1, \\ \mathbf{B}_{2\text{ref}}(x_0) & \text{if } x \in V_2 \end{cases}$$



Claim: The relative helicity of $\mathbf{B}$ in the volume V with respect to the reference field $\mathbf{B}_{\text{ref}}$ satisfies the following:

$$H_R(\mathbf{B};\mathbf{B}_{\text{ref}}) = H_R(\mathbf{B}_1;\mathbf{B}_{1\text{ref}}) + H_R(\mathbf{B}_2;\mathbf{B}_{2\text{ref}})$$

Relative Helicity and the Calabi Invariant

Set up an automorphism of a disk given by

$$\psi := \phi_{1\mathrm{ref}}^{-1} \circ \phi_{1\mathrm{ref}}^{-1} \circ \phi_2 \circ \phi_1 : D \to D$$

We aim to show that

$$\mathrm{Cal}(\psi) := H_R(\mathbf{B}, \mathbf{B}_{\mathrm{ref}}) = H_R(\mathbf{B}_1, \mathbf{B}_{1\mathrm{ref}}) + H_R(\mathbf{B}, \mathbf{B}_{\mathrm{ref}})$$
$$(= \mathrm{Cal}(\phi_{1\mathrm{ref}}^{-1} \circ \phi_1) + \mathrm{Cal}(\phi_{2\mathrm{ref}} \circ \phi_2))$$

It then remains to show that $\mathrm{Cal}(\psi) = \mathrm{Cal}(\phi_{2\mathrm{ref}}^{-1} \circ \phi_2) + \mathrm{Cal}(\phi_{1\mathrm{ref}}^{-1} \circ \phi_1)$:

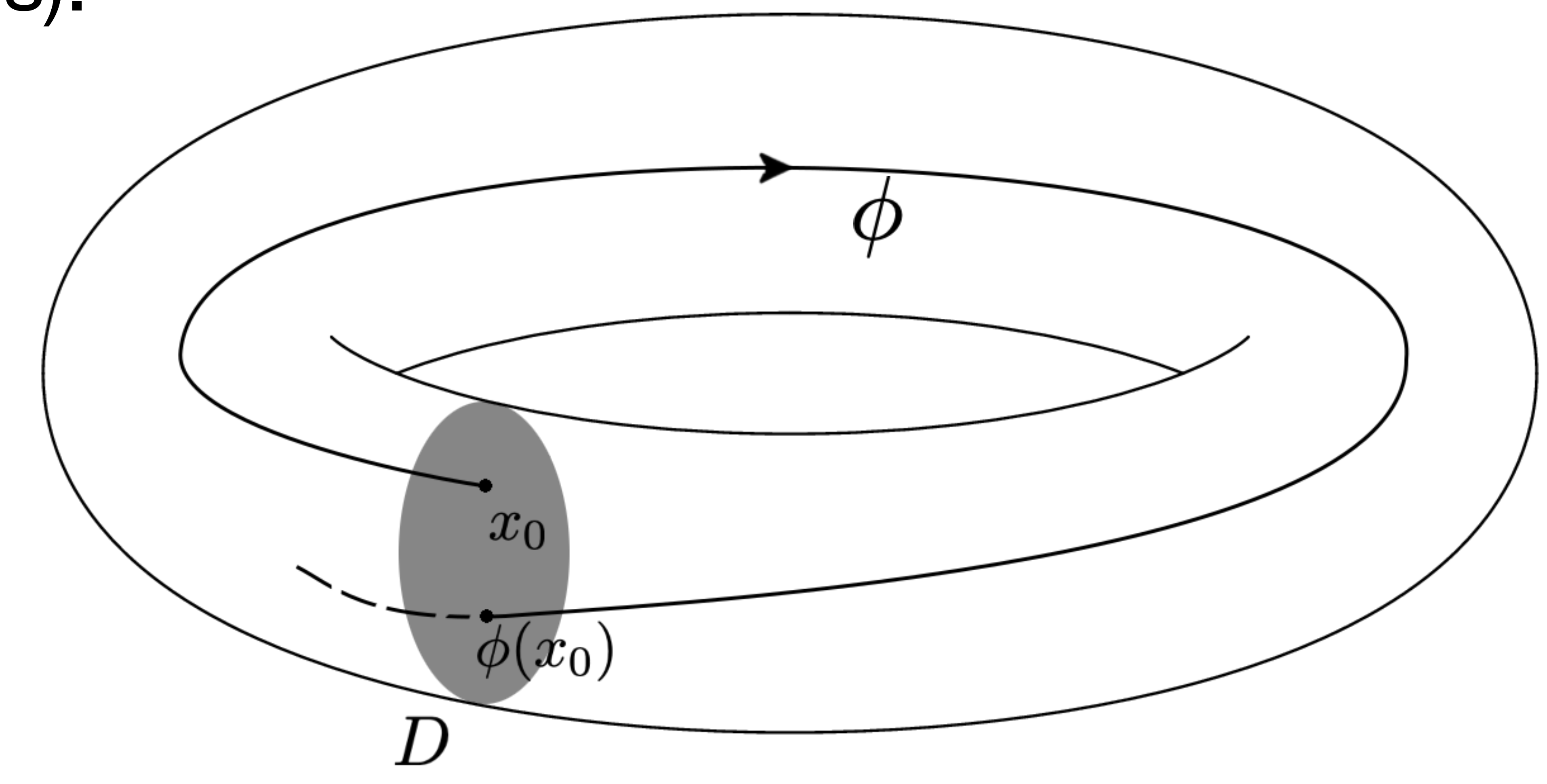
$$\begin{aligned}
\mathrm{Cal}(\psi) &= \mathrm{Cal}(\phi_{1\mathrm{ref}}^{-1} \circ \phi_{2\mathrm{ref}}^{-1} \circ \phi_2 \circ \phi_1) \\
&= \mathrm{Cal}(\phi_{1\mathrm{ref}}^{-1} \circ \phi_{2\mathrm{ref}}^{-1} \circ \phi_2 \circ \phi_{1\mathrm{ref}} \circ \phi_{1\mathrm{ref}}^{-1} \circ \phi_1) \\
&= \mathrm{Cal}(\phi_{1\mathrm{ref}}^{-1} \circ \phi_{2\mathrm{ref}}^{-1} \circ \phi_2 \circ \phi_{1\mathrm{ref}}) + \mathrm{Cal}(\phi_{1\mathrm{ref}}^{-1} \circ \phi_1) \\
&= \mathrm{Cal}(\phi_{2\mathrm{ref}}^{-1} \circ \phi_2) + \mathrm{Cal}(\phi_{1\mathrm{ref}}^{-1} \circ \phi_1)
\end{aligned}$$

Asymptotic Field Line Helicity

In general, field line helicity is only gauge invariant for periodic field lines (and in that case, only for certain periods).

To overcome this one can introduce an 'asymptotic' version of the field line helicity.

We have to calculate the limit of the field line helicity after n-many iterations.



Asymptotic Field Line Helicity

Define an iterative map $\phi^{(n)} : D \mapsto D$ and an associated field line helicity $\mathcal{A}^{(n)}$

where

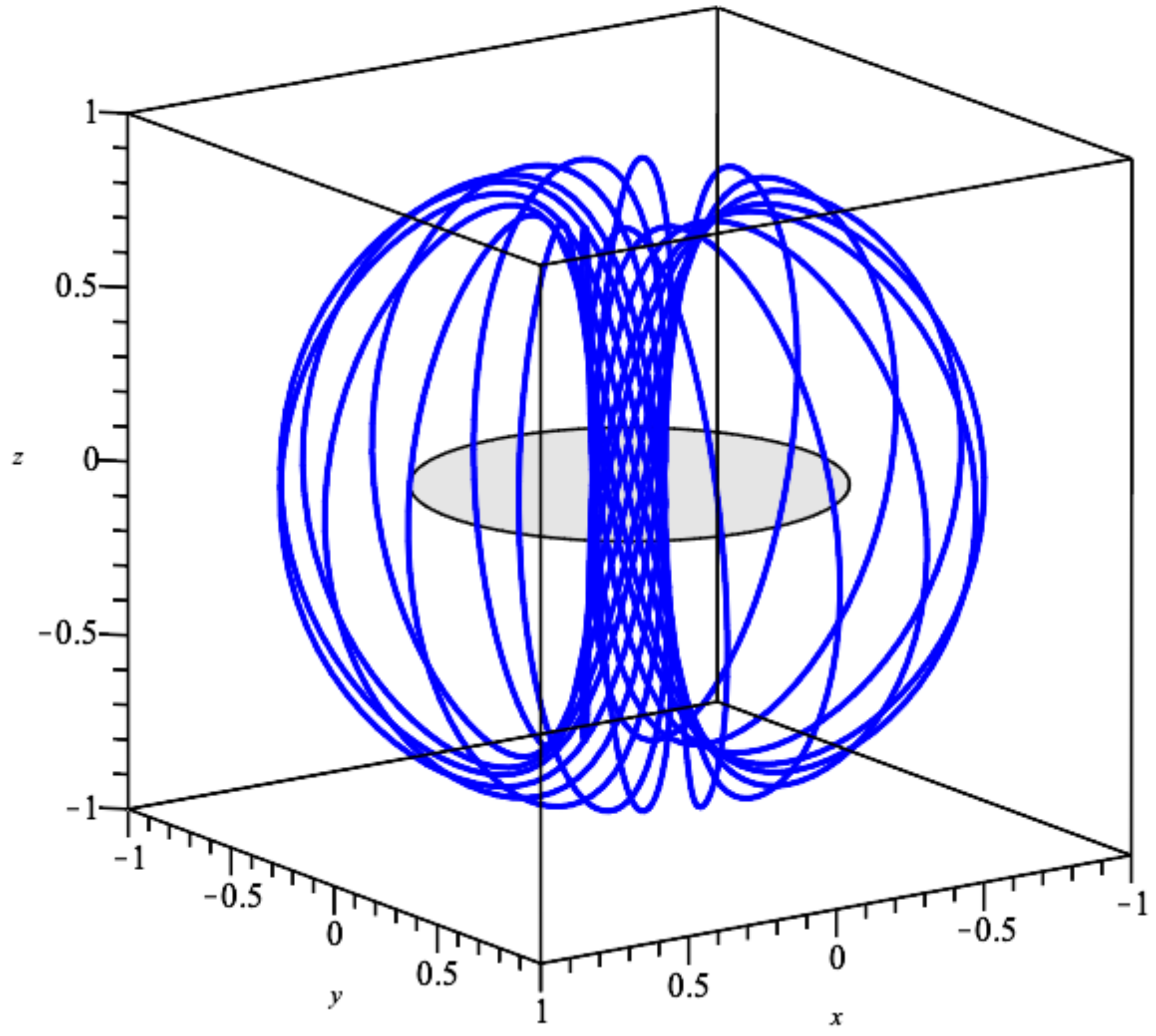
$$\mathcal{A}^{(n)}(x_0) = \sum_{k=0}^{n-1} \mathcal{A}(\phi^{(k)}(x_0)).$$

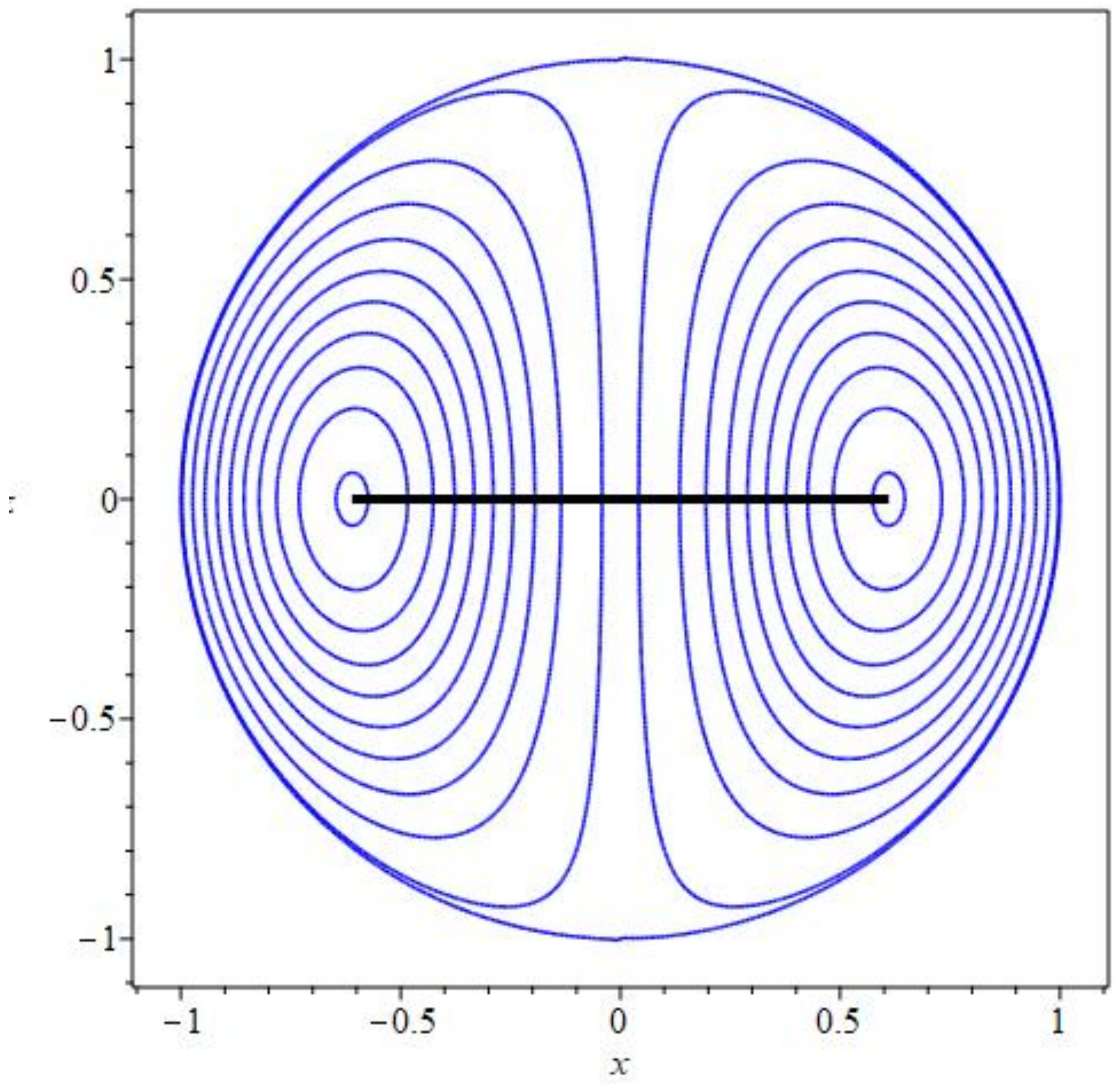
If we let $n \to \infty$ then we can define an averaged field line helicity in the asymptotic limit

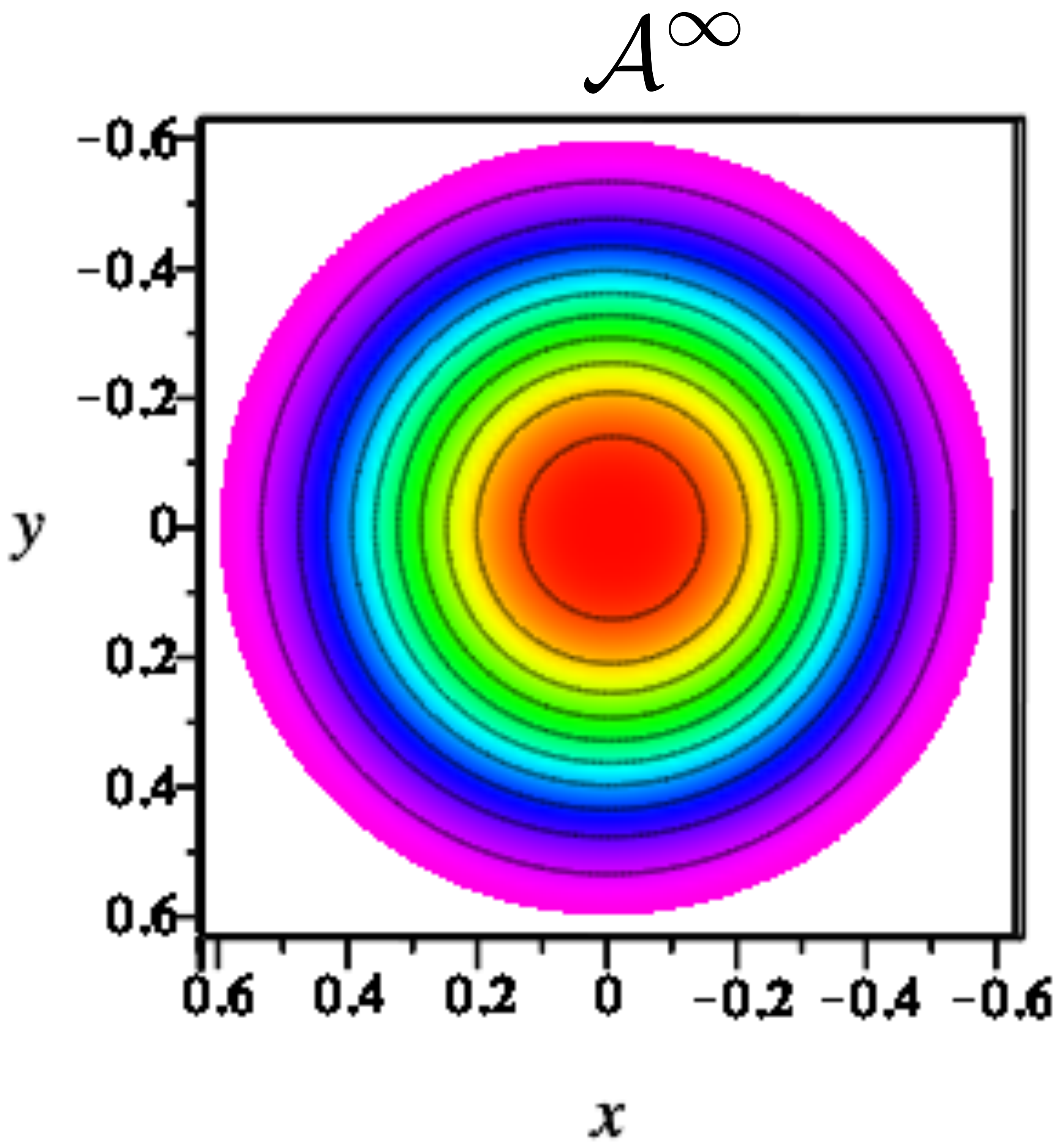
$$\bar{\mathcal{A}}^{\infty}(x_0) := \lim_{n\to\infty} \frac{1}{n}\mathcal{A}^{(n)}(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{A}(\phi^{(k)}(x_0))$$

In particular, the asymptotic field line helicity $\bar{\mathcal{A}}^{\infty}(x_0)$ is gauge invariant.

Application: Spherical force-free field







Field lines wind on tori, hence the asymptotic field line helicity is constant in ϕ-direction on the global cross section (disk).

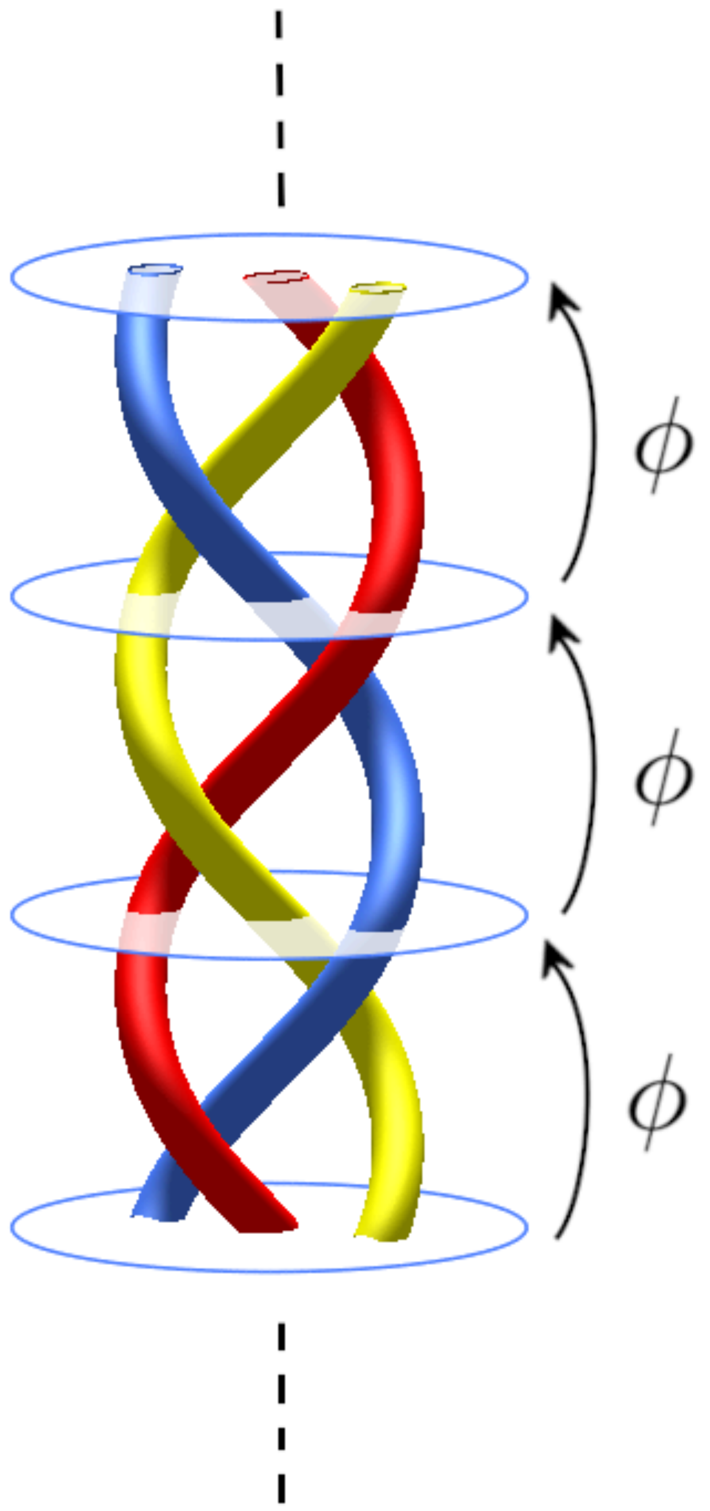
Application: Iterative Braiding

Introduce a braid-like map that introduces left handed and right handed twist in equal measure.

Classical helicity should measure a total of zero winding for such a field configuration.

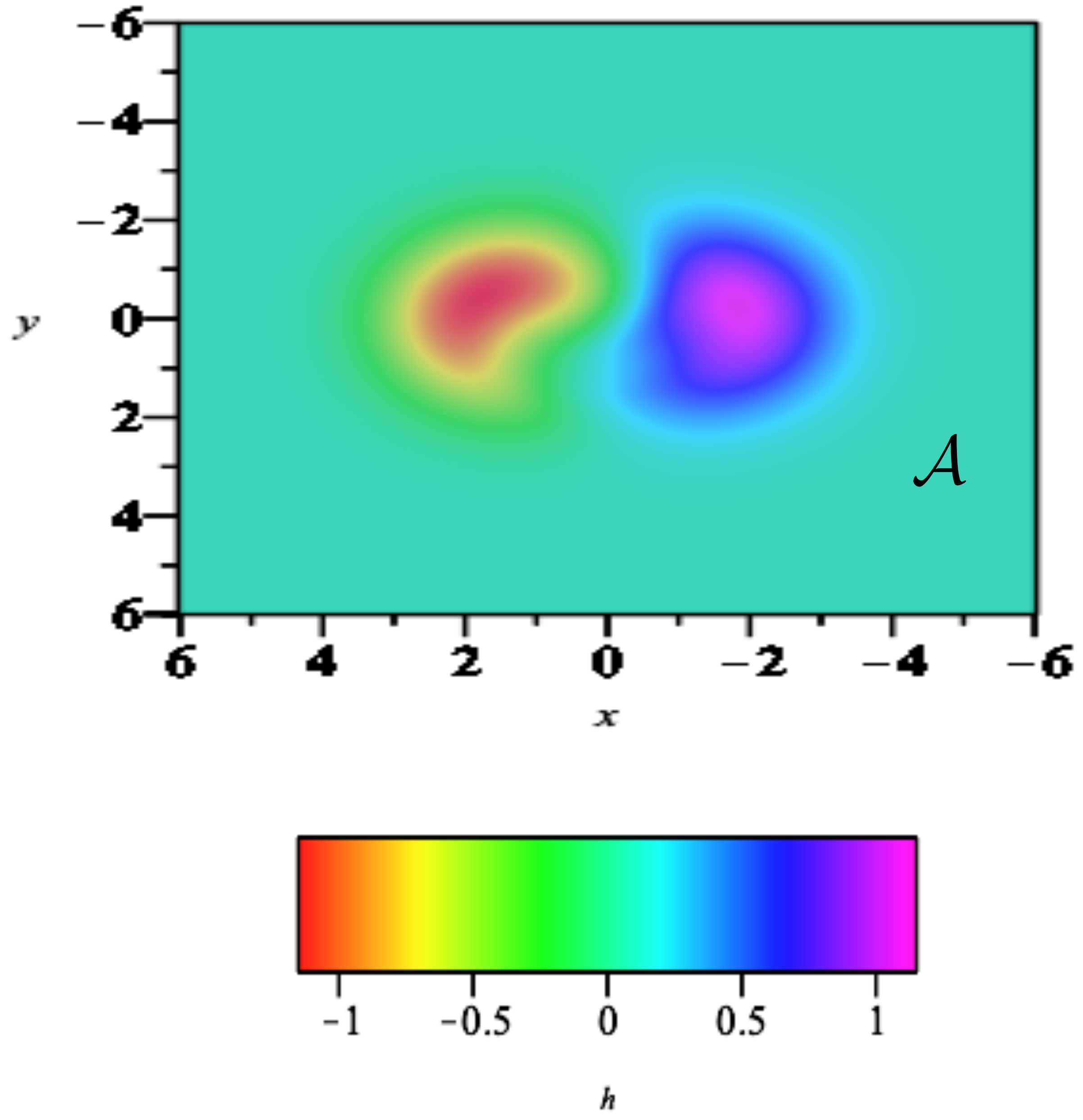
We attempt to calculate the winding across a cross-sectional disk after a large number of iterations.

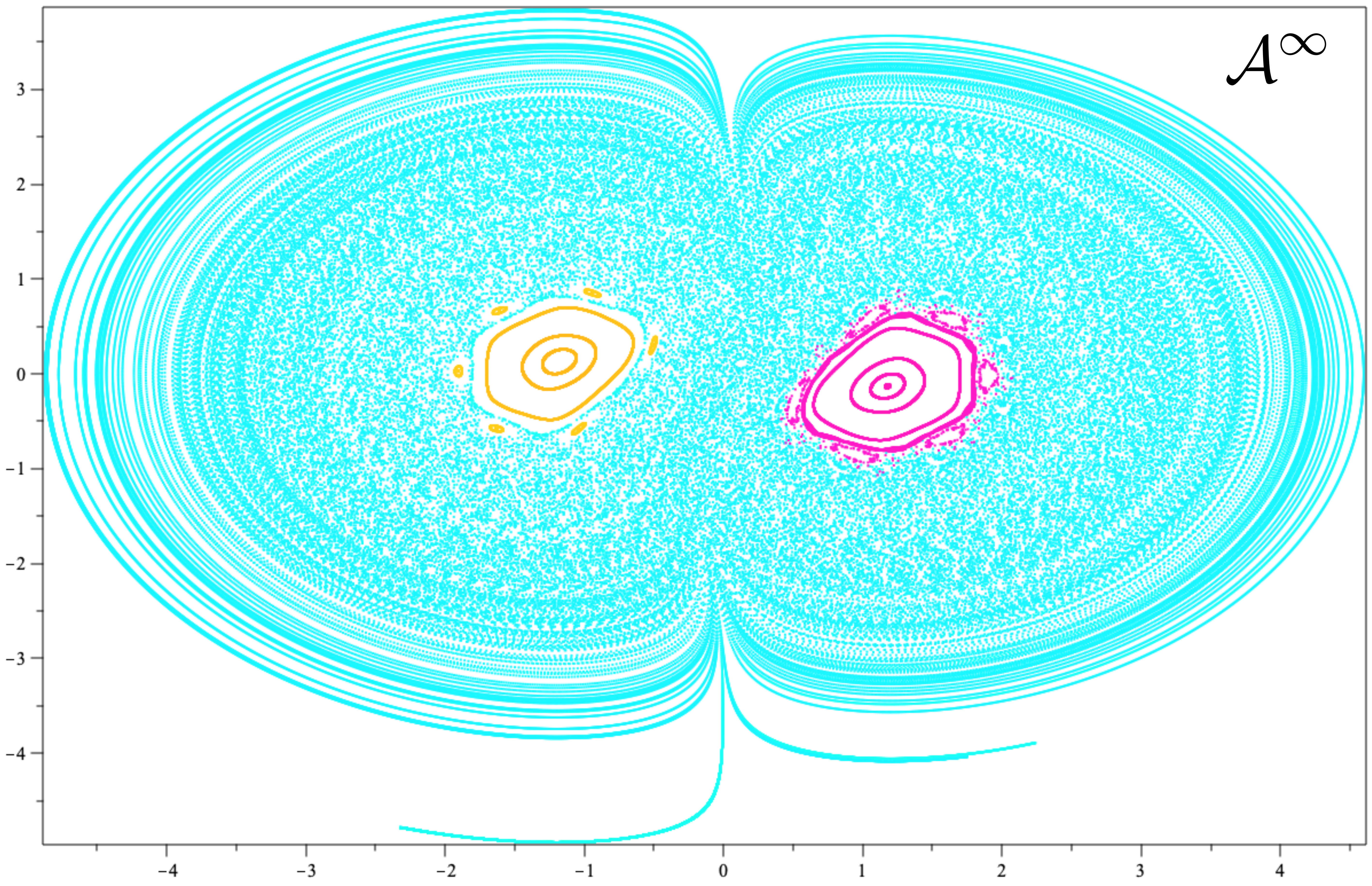
In the process we pick out local winding structures missed by the classical helicity.



Application: Iterative Braiding

Magnetic braid set up such that it has ergodic and regular regions.





Summary

- The Calabi invariant can be used to calculate helicity.
- Obtained a general formula for the field line helicity over successive iterations of a braid.
- We can construct a gauge invariant field line helicity in the asymptotic limit of an iterative field line mapping.
- We can investigate local structures of winding configurations not picked up by a classical helicity integral.

References

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