Helicity 2020: Online Advanced Study Program
on Helicities in Astrophysics and Beyond
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Unappreciated helicity effects in hydrodynamic and magnetohydrodynamic turbulence

Nobumitsu Yokoi

Institute of Industrial Science, University of Tokyo

Topics

- **Theory for inhomogeneous turbulence**

  How to tackle strongly nonlinear and inhomogeneous/anisotropic turbulence

- **Dynamo coupled with large-scale flow**

  Cross helicity effect in dynamo

- **Global flow generation due to helicities**

  Helicity and cross-helicity effects in momentum transport

How to tackle strongly nonlinear and inhomogeneous/anisotropic turbulence

Equation of fluctuating velocity $\mathbf{u} = \mathbf{U} + \mathbf{u}', \quad \mathbf{U} = \langle \mathbf{u} \rangle, \quad \mathbf{u}' = \mathbf{u} - \langle \mathbf{u} \rangle$

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

turbulence–mean velocity interaction

turbulence–turbulence interaction

Instability approach

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - \frac{\partial p'^{(\mathrm{R})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Linear in $\mathbf{u}'$ and $p'^{(\mathrm{R})}$, each (Fourier) mode evolves independently

Closure approach

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'^{(\mathrm{S})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Homogeneous turbulence, no dependence on large-scale inhomogeneity

Homogeneous turbulence

Navier–Stokes equation in the wave-number space

$$ik_a \hat{u}_a(\mathbf{k};t) = 0$$

$$\frac{\partial \hat{u}_\alpha(\mathbf{k};t)}{\partial t} - ik_a \iint d\mathbf{p} d\mathbf{q} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \hat{u}_a(\mathbf{p};t) \hat{u}_\alpha(\mathbf{q};t)$$
$$= ik_\alpha \hat{p}(\mathbf{k};t) - \nu k^2 \hat{u}_\alpha(\mathbf{k};t)$$

$$\hat{p}(\mathbf{k};t) = -\frac{k_a k_b}{k^2} \iint d\mathbf{p} d\mathbf{q} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \hat{u}_a(\mathbf{p};t) \hat{u}_b(\mathbf{q};t)$$

$$\longrightarrow \quad \frac{\partial \hat{u}_\alpha(\mathbf{k};t)}{\partial t} = -\nu k^2 \hat{u}_\alpha(\mathbf{k};t) + iM_{\alpha ab}(\mathbf{k}) \iint d\mathbf{p} d\mathbf{q} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \hat{u}_a(\mathbf{p};t) \hat{u}_b(\mathbf{q};t)$$

where $M_{\alpha ab}(\mathbf{k}) = \frac{1}{2}\left[k_b D_{\alpha a}(\mathbf{k}) + k_a D_{\alpha b}(\mathbf{k})\right]$

with the projection operator $D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$

Renormalized perturbation expansion theory

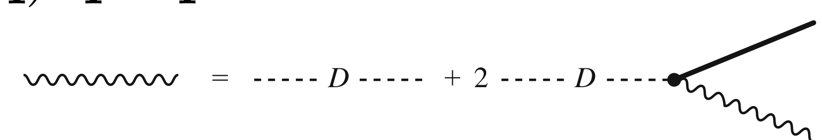
Kraichnan, R. H. (1959) “The structure of isotropic turbulence **at very high Reynolds number**,” J. Fluid Mech. **5**, 497

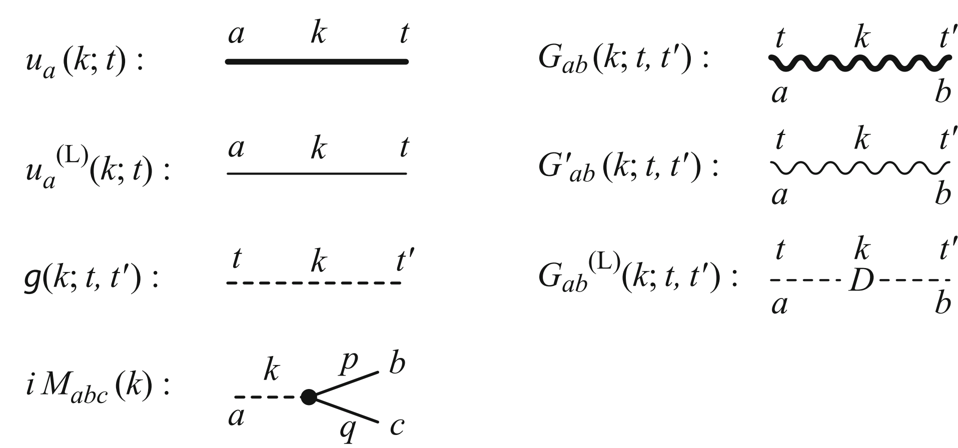
Velocity

$$u_\alpha(\mathbf{k};t) = u_\alpha^{(\mathrm{L})}(\mathbf{k};t) + iM_{cab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q}$$
$$\times \int_{-\infty}^{t} dt_1 G_{\alpha c}^{(\mathrm{L})}(\mathbf{k};t,t_1) u_a(\mathbf{p};t_1) u_b(\mathbf{q};t_1)$$

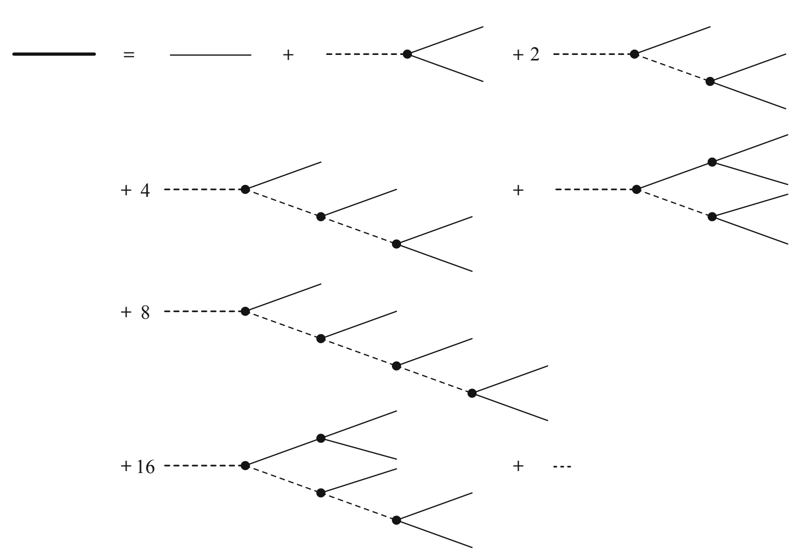
Response function

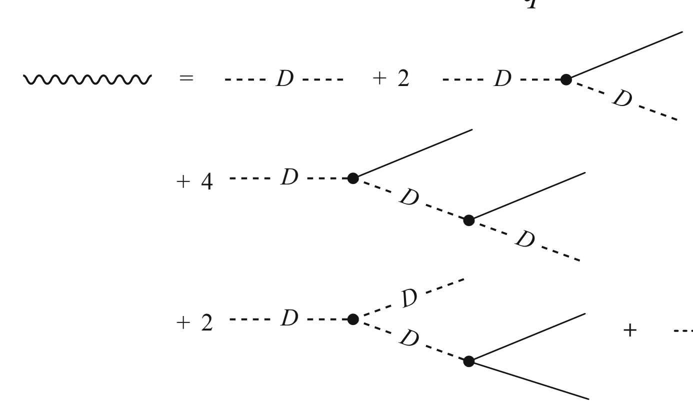
$$G'_{\alpha\beta}(\mathbf{k};t,t') = G_{\alpha\beta}^{(\mathrm{L})}(\mathbf{k};t,t') + iM_{cab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q}$$
$$\times \int_{t'}^{t} dt_1 G_{\alpha c}^{(\mathrm{L})}(\mathbf{k};t,t_1) u_a(\mathbf{p};t_1) G'_{b\beta}(\mathbf{q};t,t_1)$$



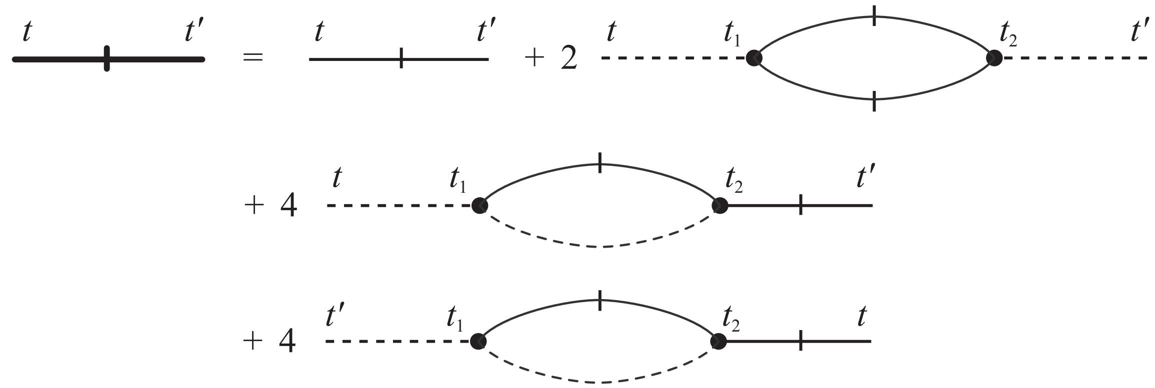


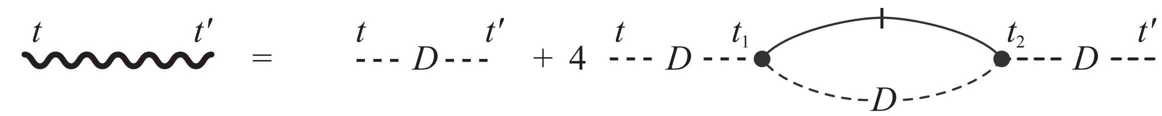
Perturbation expansion





Correlation function $Q_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$ Response function $G_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$





Renormalization

a part of the infinite series with respect to the propagators is **summed up to the infinite orders**

$$f_{\mathrm{ex}}(x) = 1 + x + x^2 + x^3 + \cdots$$

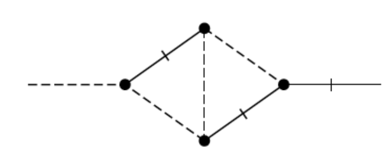
$$f_{\mathrm{ex}}(x) = 1 + x(1 + x + x^2 + \cdots)$$

Truncation $f_{\mathrm{ex}}(x) = 1 + x(1 + x)$ $\times$

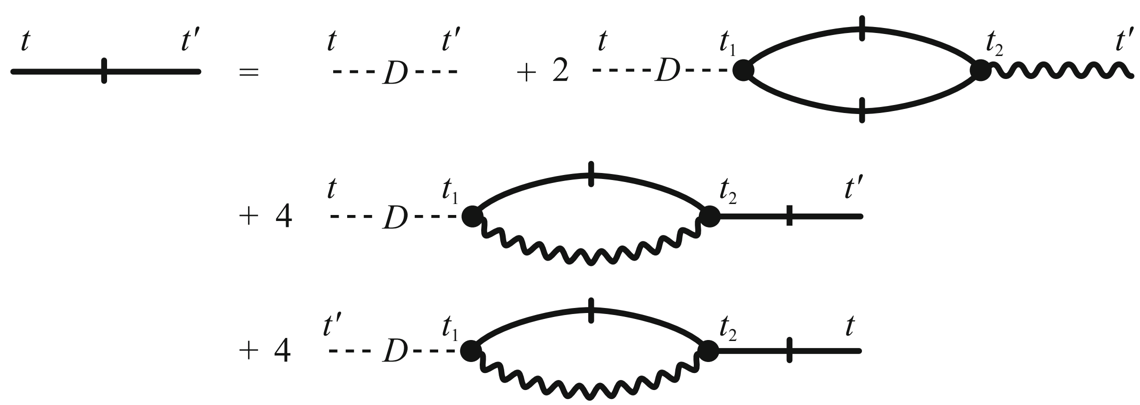
Renormalization $f_{\mathrm{ex}}(x) = 1 + x f_{\mathrm{ex}}(x)$

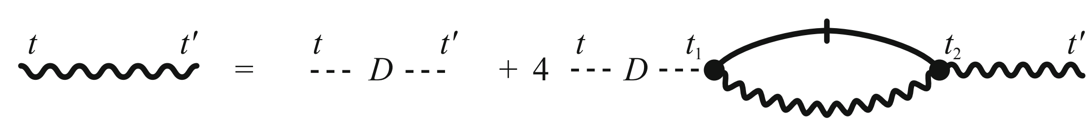
$$f_{\mathrm{ex}}(x) = \frac{1}{1-x}$$

DIA = line (propagator) renormalization (lowest-order in vertex)



not included





Multiple-scale analysis

mirror-symmetric case: Yoshizawa, Phys. Fluids **27**, 1377 (1984)

non-mirror-symmetric case: Yokoi & Yoshizawa, Phys. Fluids A **5**, 464 (1993)

Fast and slow variables

$$\boldsymbol{\xi} = \mathbf{x},\ \mathbf{X} = \delta \mathbf{x};\ \tau = t,\ T = \delta t$$

Slow variables ***X*** and *T* change only when ***x*** and *t* change much.

$$f = F\left(\mathbf{X}; T\right) + f'\left(\boldsymbol{\xi}, \mathbf{X}; \tau, T\right)$$

$$\nabla = \nabla_{\boldsymbol{\xi}} + \delta \nabla_{\mathbf{X}};\quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \delta \frac{\partial}{\partial T}$$

Velocity-fluctuation equation

$$\frac{\partial u'_\alpha}{\partial \tau} + U_a \frac{\partial u'_\alpha}{\partial \xi_a} + \frac{\partial}{\partial \xi_a} u'_a u'_\alpha + \frac{\partial p'}{\partial \xi_\alpha} - \nu \nabla_\xi^2 u'_\alpha$$
$$= \delta \left(-u'_a \frac{\partial U_\alpha}{\partial X_a} - \frac{D u'_\alpha}{DT} - \frac{\partial p'}{\partial X_\alpha} - \frac{\partial}{\partial X_a} \left(u'_a u'_\alpha - R_{a\alpha} + 2\nu \frac{\partial^2 u'_\alpha}{\partial X_a \partial \xi_a} \right) \right)$$
$$+ \delta^2 \left(\nu \nabla_X^2 u'_\alpha \right)$$

$$\frac{\partial u'_a}{\partial \xi_a} + \delta \frac{\partial u'_a}{\partial X_a} = 0$$

where $\displaystyle \frac{D}{DT} = \frac{\partial}{\partial T} + \mathbf{U} \cdot \nabla_X$

Scale parameter expansion

$$f' = f'_0 + \delta f'_1 + \delta^2 f'_2 + \cdots = \sum_n \delta^n f'_n$$

Response function

$$\frac{\partial G'_{\alpha\beta}(\mathbf{k};\tau,\tau')}{\partial\tau} + \nu k^2 G'_{\alpha\beta}(\mathbf{k};\tau,\tau')$$
$$-2iM^{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) G'_{b\beta}(\mathbf{q};\tau,\tau')$$
$$= D_{\alpha\beta}(\mathbf{k})\delta(\tau-\tau')$$

1st-order field

$$\frac{\partial u'_{1\alpha}(\mathbf{k};\tau)}{\partial\tau} + \nu k^2 u'_{1\alpha}(\mathbf{k};\tau)$$
$$-2iM_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) u'_{S1b}(\mathbf{q};\tau)$$
$$= -D_{\alpha b}(\mathbf{k}) u'_{0a}(\mathbf{k};\tau) \frac{\partial U_b}{\partial X_a} - D_{\alpha a}(\mathbf{k}) \frac{D u'_{0a}(\mathbf{k};\tau)}{D T_{\mathrm{I}}}$$
$$+2M_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau) \frac{\partial u'_{0c}(\mathbf{q};\tau)}{\partial X_{\mathrm{I}c}}$$
$$-D_{\alpha d}(\mathbf{k}) M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau) u'_{0b}(\mathbf{q};\tau)\right)$$

$$\mathbf{u}'_1(\mathbf{k};\tau) = \mathbf{u}'_{\mathrm{S}1}(\mathbf{k};\tau) - i\frac{\mathbf{k}}{k^2}\frac{\partial u'_{0a}}{\partial X_{\mathrm{I}a}} \qquad \mathbf{k}\cdot\mathbf{u}'_{\mathrm{S}1}(\mathbf{k};\tau) = 0$$

Formal solution in terms of $G'_{\alpha\beta}(\mathbf{k};\tau,\tau')$

$$
\begin{aligned}
u'_{\mathrm{S}1\alpha}(\mathbf{k};\tau) = & -\frac{\partial U_b}{\partial X_a} \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha b}(\mathbf{k};\tau,\tau_1) u'_{0a}(\mathbf{k};\tau_1) \\
& - \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha a}(\mathbf{k};\tau,\tau_1) \frac{D u'_{0a}(\mathbf{k};\tau_1)}{D T_{\mathrm{I}}} \\
& + 2M_{dab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \\
& \qquad \times \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau_1) \frac{\partial u'_{0c}(\mathbf{q};\tau_1)}{\partial X_{\mathrm{I}c}} \\
& - M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \\
& \qquad \times \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau_1) u'_{0b}(\mathbf{q};\tau_1) \right)
\end{aligned}
$$

$u'_{1\alpha}(\mathbf{k};t) = \cdots$ in terms of the force terms (r.h.s.) and response functions

Calculation of turbulent correlations with DIA

$$
\begin{aligned}
\langle f'(\mathbf{x};t) g'(\mathbf{x};\mathbf{t}) \rangle &= \int d\mathbf{k} \left\langle f'(\mathbf{k};\tau) g'(\mathbf{k};\tau) \right\rangle / \delta(\mathbf{0}) \\
&= \int d\mathbf{k} \left(\langle f'_0 g'_0 \rangle + \langle f'_0 g'_1 \rangle + \langle f'_1 g'_0 \rangle + \cdots \right) / \delta(\mathbf{0})
\end{aligned}
$$

Mean-field equations in compressible MHD

Means and fluctuations

$$f = F + f', \quad F = \langle f \rangle$$

Density

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho}\mathbf{U}) = -\nabla \cdot \langle \rho' \mathbf{u}' \rangle$$

Momentum

$$\frac{\partial}{\partial t}\overline{\rho}U^\alpha + \frac{\partial}{\partial x^a}\overline{\rho}U^a U^\alpha$$
$$= -(\gamma_0 - 1)\frac{\partial}{\partial x^\alpha}\overline{\rho}Q + \frac{\partial}{\partial x^\alpha}\mu S^{a\alpha} + (\mathbf{J}\times\mathbf{B})^\alpha$$
$$- \frac{\partial}{\partial x^\alpha}\left(\overline{\rho}\langle u'^a u'^\alpha\rangle - \frac{1}{\mu_0}\langle b'^a b'^\alpha\rangle + U^a\langle \rho' u'^\alpha\rangle + U^\alpha\langle \rho' u'^a\rangle\right) + R_U^\alpha$$

Internal energy

$$\frac{\partial}{\partial t}\overline{\rho}Q + \nabla \cdot (\overline{\rho}\mathbf{U}Q) = \nabla \cdot \left(\frac{\kappa}{C_V}\nabla Q\right) - \nabla \cdot (\overline{\rho}\langle q'\mathbf{u}'\rangle + Q\langle \rho'\mathbf{u}'\rangle + \mathbf{U}\langle \rho' q'\rangle)$$
$$-(\gamma_0 - 1)\left(\overline{\rho}Q\nabla \cdot \mathbf{U} + \overline{\rho}\langle q'\nabla \cdot \mathbf{u}'\rangle + Q\langle \rho'\nabla \cdot \mathbf{u}'\rangle\right) + R_Q$$

Magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U}\times\mathbf{B} + \langle \mathbf{u}' \times \mathbf{b}'\rangle) + \eta\nabla^2\mathbf{B}$$

where

$$R_U^\alpha = -\frac{\partial}{\partial t}\langle \rho' u'^\alpha\rangle - \frac{\partial}{\partial x^a}\langle \rho' u'^a u'^\alpha\rangle$$
$$-(\gamma_0 - 1)\frac{\partial}{\partial x^\alpha}\langle \rho' q'\rangle - \frac{1}{2\mu_0}\frac{\partial}{\partial x^\alpha}\langle \mathbf{b}'^2\rangle \qquad \text{etc.}$$

Statistical assumptions on the lowest-order (basic) fields

Basic fields are homogeneous isotropic

$$\frac{\langle \rho'_{\rm B}(\mathbf{k};\tau)\rho'_{\rm B}(\mathbf{k}';\tau')\rangle}{\delta(\mathbf{k}+\mathbf{k}')} = \left\langle Q'_\rho(k;\tau,\tau')\right\rangle = Q_\rho(k;\tau,\tau'),$$

$$\frac{\langle {\vartheta'_{\rm B}}^\alpha(\mathbf{k};\tau){\chi'_{\rm B}}^\beta(\mathbf{k}';\tau')\rangle}{\delta(\mathbf{k}+\mathbf{k}')}$$

$$= D^{\alpha\beta}(\mathbf{k})Q_{\vartheta\chi{\rm S}}(k;\tau,\tau') + \Pi^{\alpha\beta}(\mathbf{k})Q_{\vartheta\chi{\rm C}}(k;\tau,\tau') + \frac{i}{2}\frac{k^c}{k^2}\epsilon^{\alpha\beta c}H_{\vartheta\chi}(k;\tau,\tau')$$

$$\frac{\langle q'_{\rm B}(\mathbf{k};\tau)q'_{\rm B}(\mathbf{k}';\tau')\rangle}{\delta(\mathbf{k}+\mathbf{k}')} = \left\langle Q'_q(k;\tau,\tau')\right\rangle = Q_q(k;\tau,\tau'),$$

with solenoidal and dilatational projection operators

$$D^{\alpha\beta}(\mathbf{k}) = \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}, \quad \Pi^{\alpha\beta}(\mathbf{k}) = \frac{k^\alpha k^\beta}{k^2}$$

Dynamo coupled with large-scale flows:

Cross-helicity effect in dynamo

Modelling in dynamos

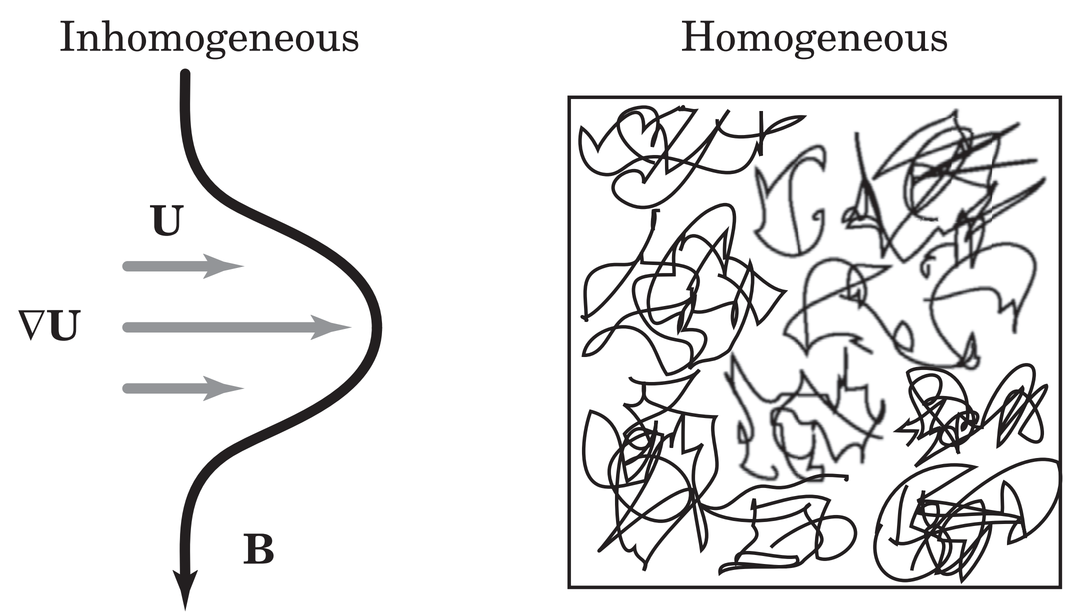
$$\langle \mathbf{u}' \times \mathbf{b}' \rangle^\alpha = \alpha^{\alpha a} B^a + \beta^{\alpha a b} \frac{\partial B^a}{\partial x^b} + \cdots$$

Mean field

$$\mathbf{b} = \mathbf{B} + \mathbf{b}', \quad \mathbf{B} = \langle \mathbf{b} \rangle$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \langle \mathbf{u}' \times \mathbf{b}' \rangle + \eta \nabla^2 \mathbf{B}$$

$(\mathbf{B} \cdot \nabla)\mathbf{U}$ ⟶ differential rotation, “Ω effect”



Turbulence

$$\mathbf{U} = \mathbf{U}_0(\text{constant}) \;\; \text{or} \;\; \mathbf{0}$$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u}' = (\mathbf{B} \cdot \nabla)\mathbf{b}' + (\mathbf{b}' \cdot \nabla)\mathbf{B} - (\mathbf{u}' \cdot \nabla)\mathbf{U} + \cdots$$

$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{b}' = (\mathbf{B} \cdot \nabla)\mathbf{u}' - (\mathbf{u}' \cdot \nabla)\mathbf{B} + (\mathbf{b}' \cdot \nabla)\mathbf{U} + \cdots$$

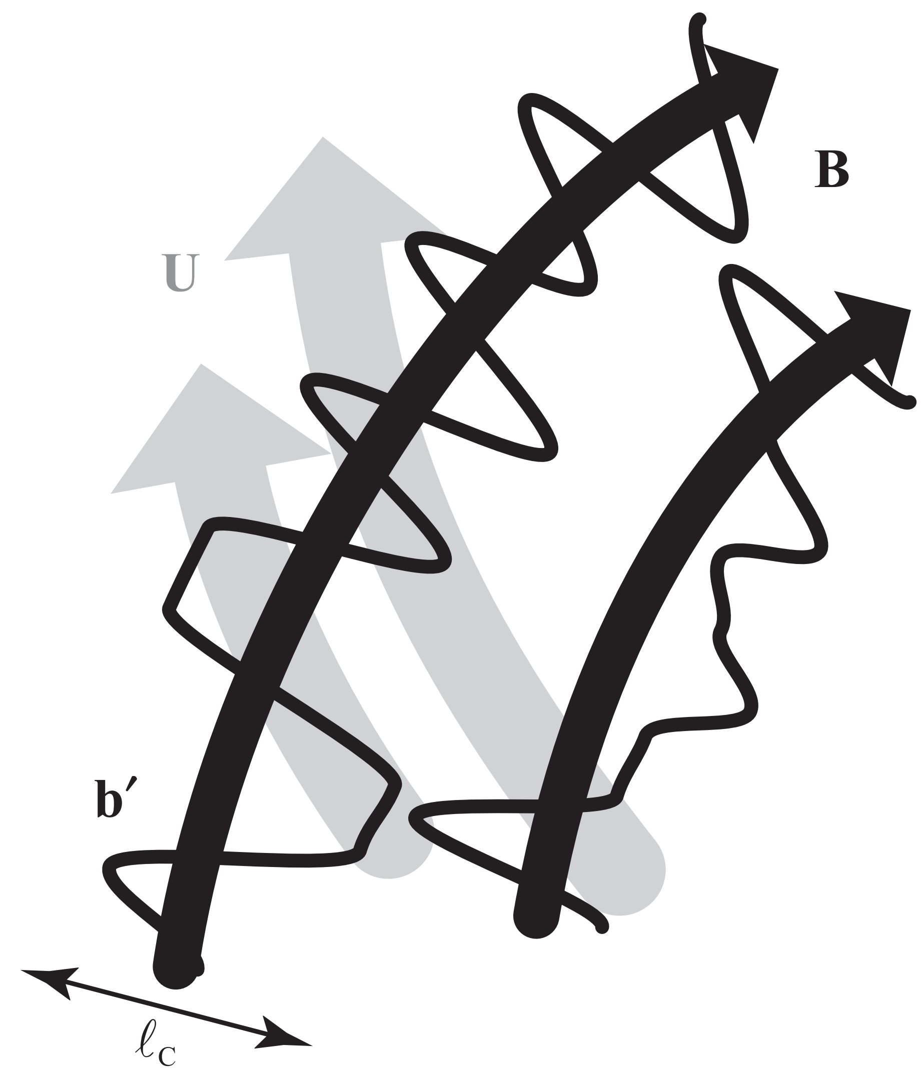
⟶ $\langle \mathbf{u}' \times \mathbf{b}' \rangle^\alpha = \alpha^{\alpha a} B^a + \beta^{\alpha a b} \frac{\partial B^a}{\partial x^b} + \cdots$ “Ansatz”

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{u}' = (\mathbf{B}\cdot\nabla)\mathbf{b}' + (\mathbf{b}'\cdot\nabla)\mathbf{B} - (\mathbf{u}'\cdot\nabla)\mathbf{U} + \cdots$$

$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{b}' = (\mathbf{B}\cdot\nabla)\mathbf{u}' - (\mathbf{u}'\cdot\nabla)\mathbf{B} + (\mathbf{b}'\cdot\nabla)\mathbf{U} + \cdots$$

$$\left\langle \frac{\partial \mathbf{u}'}{\partial t} \times \mathbf{b}' \right\rangle + \left\langle \mathbf{u}' \times \frac{\partial \mathbf{b}'}{\partial t} \right\rangle = \cdots$$

$$\begin{aligned}
&\tau\langle \mathbf{u}' \times [(\mathbf{b}'\cdot\nabla)\mathbf{U}] + [(\mathbf{u}'\cdot\nabla)\mathbf{U}] \times \mathbf{b}' \rangle^{\alpha} \\
&\quad = \epsilon^{\alpha ab}\tau\langle u'^a b'^c\rangle \frac{\partial U^b}{\partial x^c} - \epsilon^{\alpha ba}\tau\langle b'^a u'^c\rangle \frac{\partial U^b}{\partial x^c} \\
&\quad = \tau\left(\langle u'^a b'^c\rangle + \langle u'^c b'^a\rangle\right)\epsilon^{\alpha ab}\frac{\partial U^b}{\partial x^c}
\end{aligned}$$



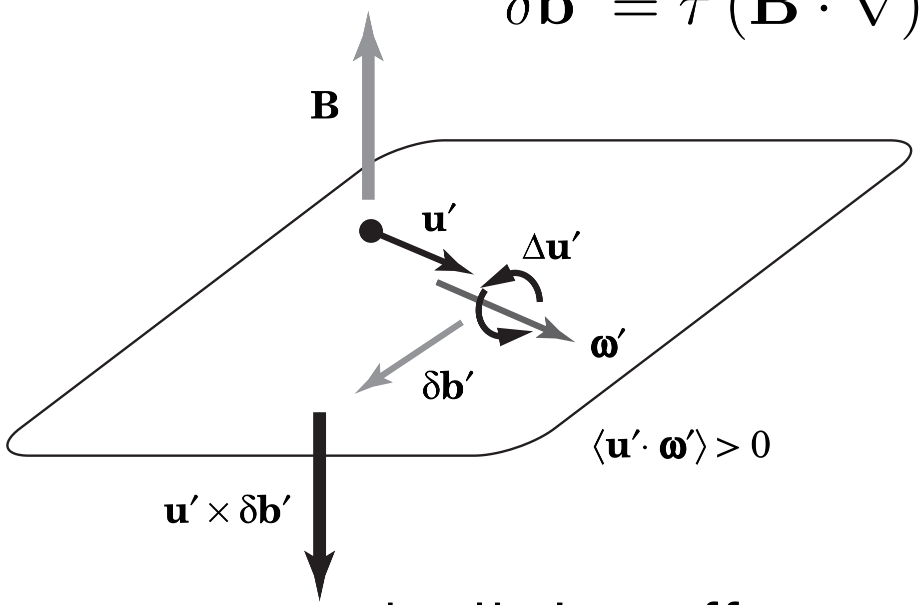
$$\longrightarrow \qquad \langle \mathbf{u}' \times \mathbf{b}' \rangle = \cdots + \tau\langle \mathbf{u}'\cdot\mathbf{b}'\rangle \nabla \times \mathbf{U} + \cdots$$

cross helicity

α and cross-helicity effects

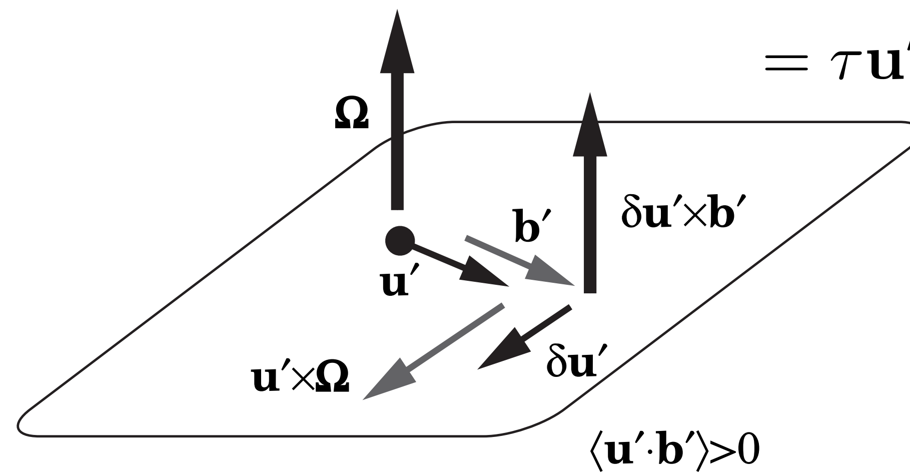
(Yokoi, GAFD **107**, 114, 2013)

$$\delta \mathbf{b}' = \tau \left(\mathbf{B} \cdot \nabla \right) \mathbf{u}' + \cdots$$



helicity effect

$$\delta \mathbf{u}' = \tau \left(\mathbf{u}' \cdot \nabla \right) \mathbf{U} + \cdots$$
$$= \tau \mathbf{u}' \times \boldsymbol{\Omega} + \cdots$$

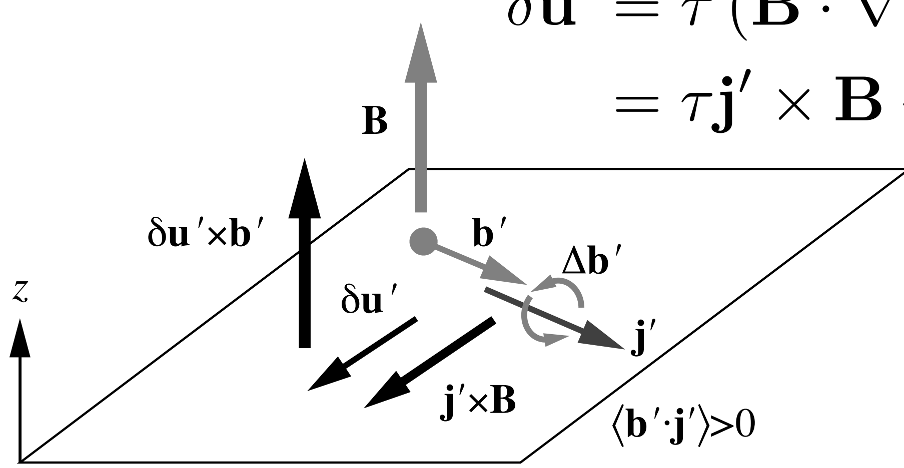


mean vorticity

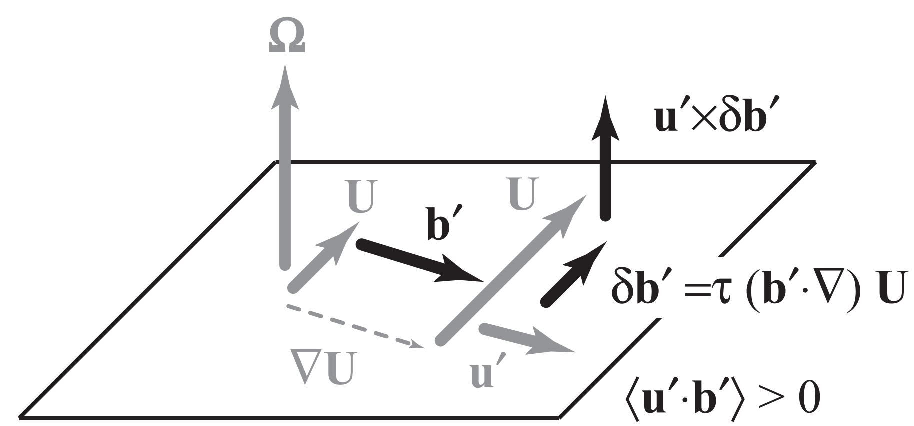
$$\boldsymbol{\Omega} = \nabla \times \mathbf{U}$$

cross-helicity effect

$$\delta \mathbf{u}' = \tau \left(\mathbf{B} \cdot \nabla \right) \mathbf{b}' + \cdots$$
$$= \tau \mathbf{j}' \times \mathbf{B} + \cdots$$



$$\delta \mathbf{b}' = \tau \left(\mathbf{b}' \cdot \nabla \right) \mathbf{U} + \cdots$$



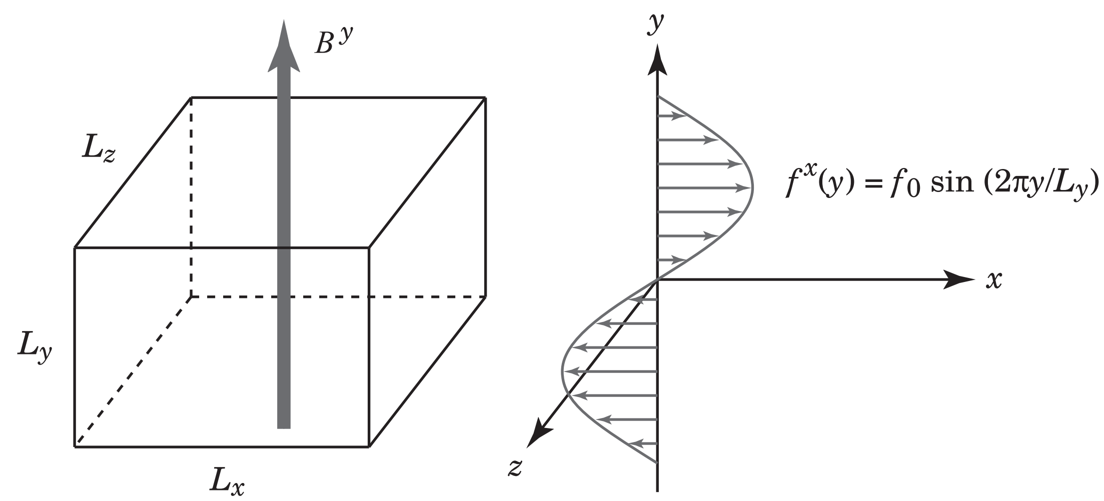
α dynamo

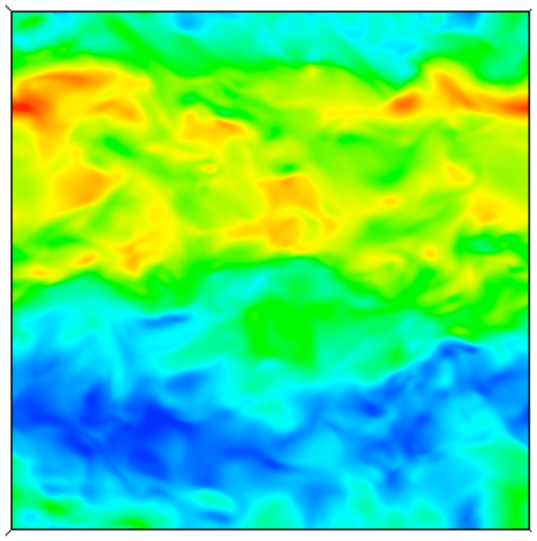
$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U}$$

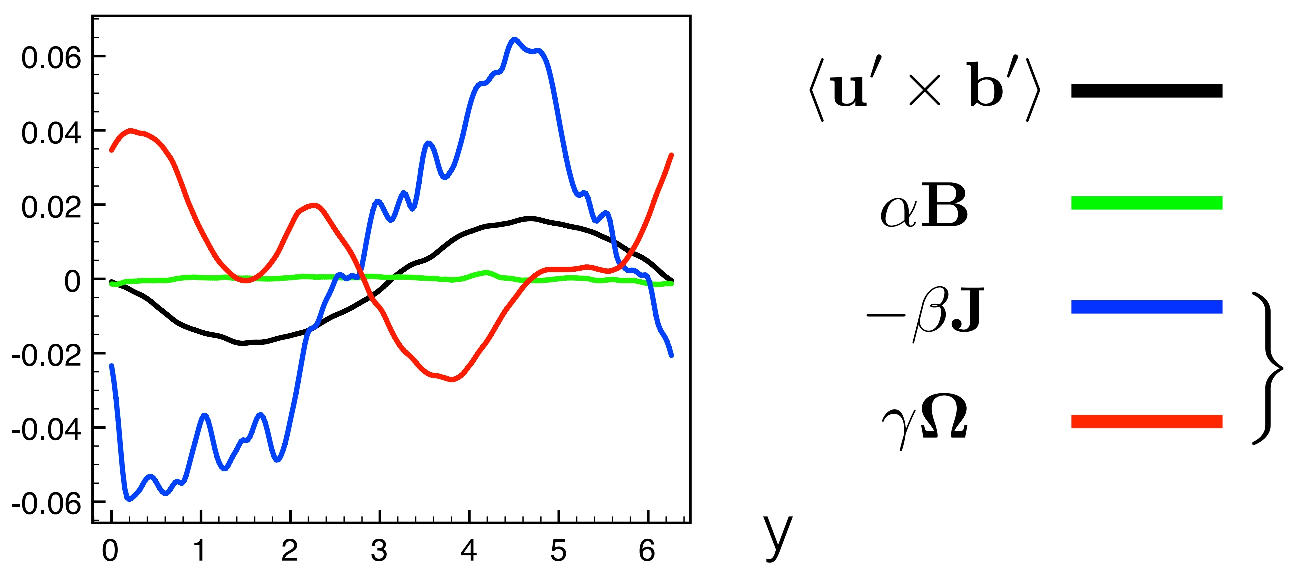
cross-helicity dynamo

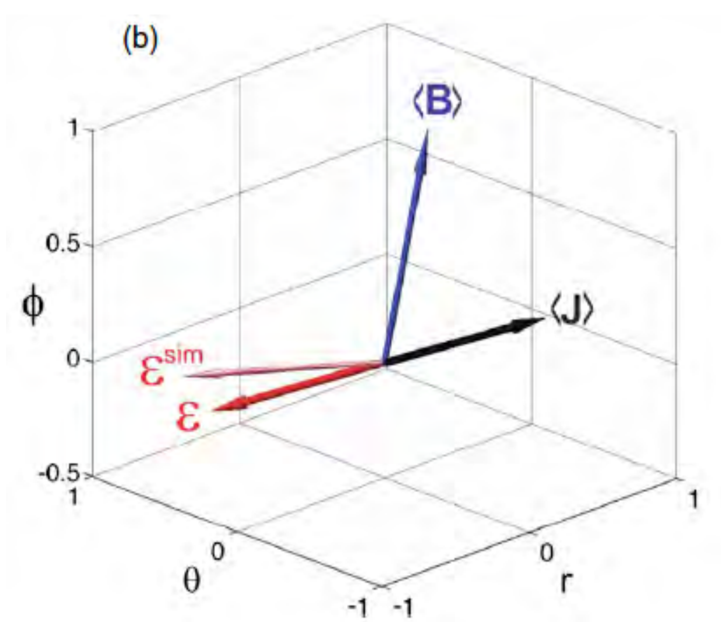
Numerical validation of cross-helicity effect

DNS of electromotive force in Kolmogorov flow (Yokoi & Balarac, 2011)







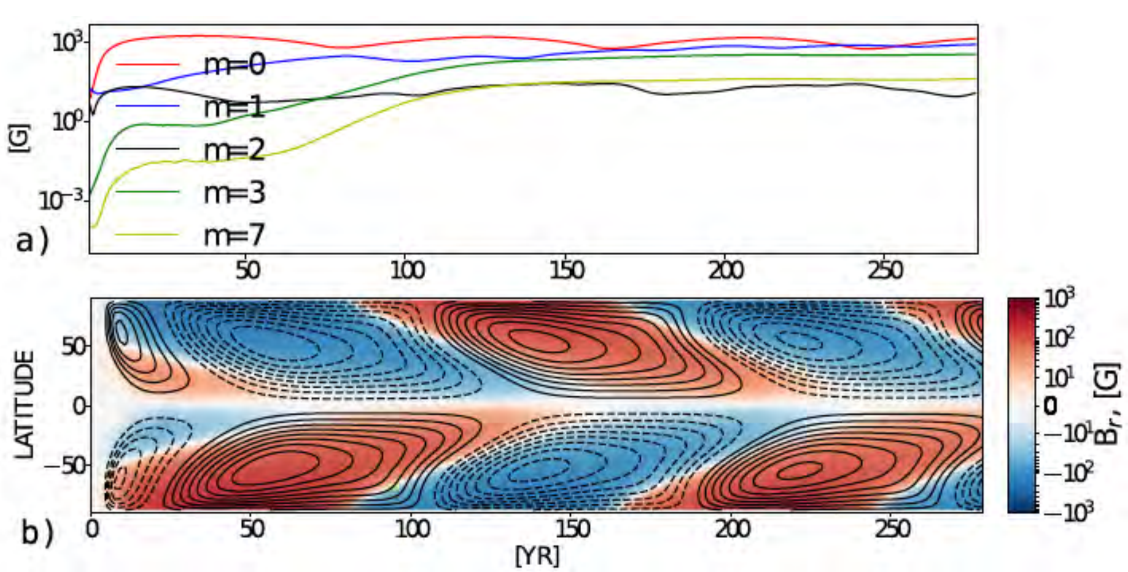


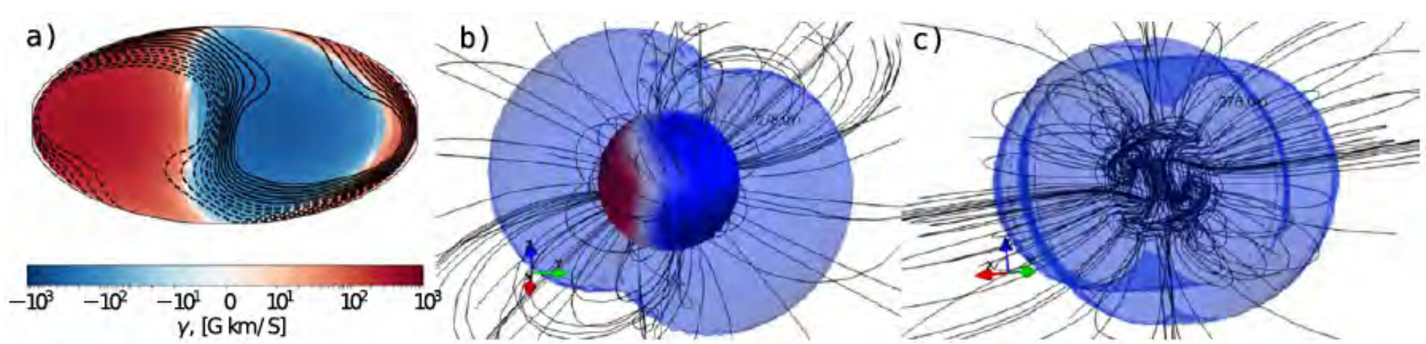
(Rahbarnia, et al. (2012) ApJ

Cross-helicity dynamo
for fully convective stars (cool stars)

Pipin & Yokoi (2018) Astrophys. J. **859**, 18

For a particular case of the fast rotating stars with solid body rotation regime, we show a possibility to sustain the strong dipolar B-field via $\alpha^2\gamma^2$ dynamo.



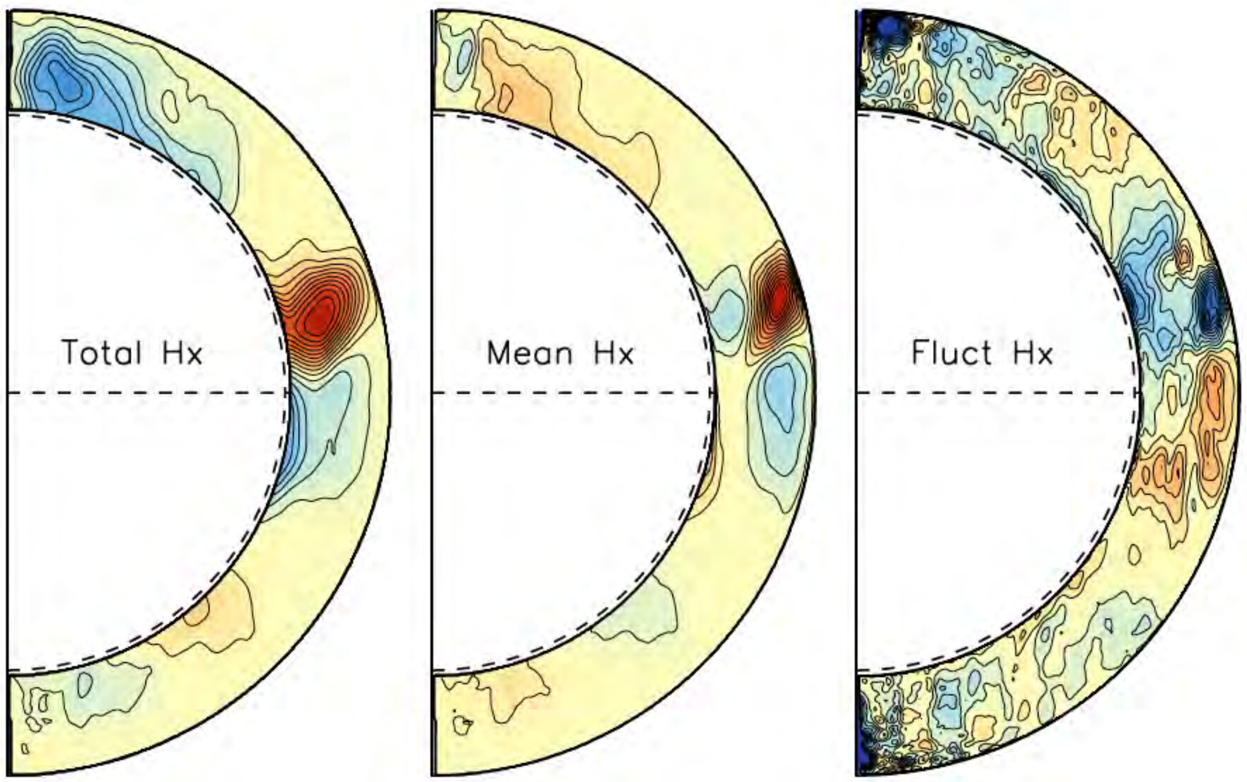


Relative importance of cross-helicity to differential-rotation effects

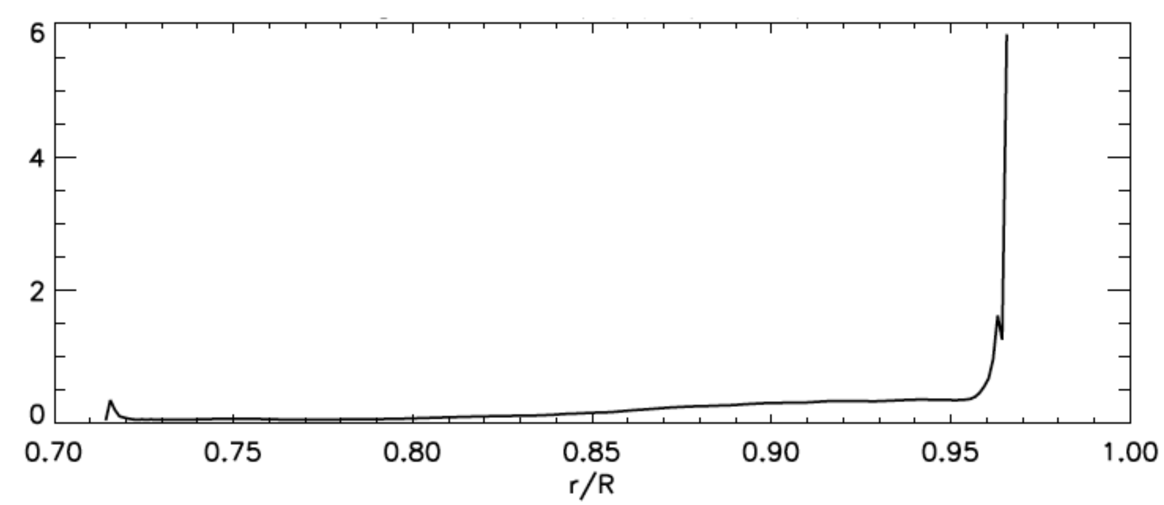
$$\frac{(\text{cross-helicity effect})}{(\text{differential-rotation effect})} = \frac{|\nabla \times (\gamma \nabla \times \mathbf{U})|}{|\nabla \times (\mathbf{U} \times \mathbf{B})|}$$

$$\sim \frac{\langle \mathbf{u}' \cdot \mathbf{b}' \rangle}{D \left(\dfrac{\partial U}{\partial r} \right) B^r} \frac{\tau_{\rm turb}}{\tau_{\rm mean}} \sim \frac{\langle \mathbf{u}' \cdot \mathbf{b}' \rangle}{\delta U B^r} Ro^{-1} = \frac{\langle \mathbf{u}' \cdot \mathbf{b}' \rangle}{\delta U B^r} \frac{K/\varepsilon}{D/\delta U}$$

Spatial distribution of cross helicity



Relative magnitude of the cross-helicity to the differential rotation terms



Provided by Mark Miesch (2016)

Global flow generation

Inhomogeneous helicity and cross helicity effects in momentum transport

Vortex generation

Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \underbrace{\frac{\nabla \rho \times \nabla p}{\rho^2}}_{\text{baroclinicity}} + \nu \nabla^2 \boldsymbol{\omega}$$

cf., Biermann battery $-\dfrac{\nabla n_e \times \nabla p_e}{n_e^2 e}$

Mean vorticity

$$\boldsymbol{\Omega} = \nabla \times \mathbf{U}$$

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\mathbf{U} \times \boldsymbol{\Omega}) + \nabla \times \underbrace{\langle \mathbf{u}' \times \boldsymbol{\omega}' \rangle}_{\mathbf{V}_{\mathrm{M}}} + \nu \nabla^2 \boldsymbol{\Omega}$$

vortexmotive force

cf., Mean magnetic field $\dfrac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \langle \mathbf{u}' \times \mathbf{b}' \rangle + \eta \nabla^2 \mathbf{B}$

electromotive force

Reynolds stress $\mathcal{R}^{ij} = \langle u'^i u'^j \rangle$ $\qquad V_{\mathrm{M}}^i = -\dfrac{\partial \mathcal{R}^{ij}}{\partial x^j} + \dfrac{\partial K}{\partial x^i}$

Theoretical formulation

Basic field: homogeneous isotropic but non-mirror-symmetric

$$\frac{\langle u'_{0\alpha}(\mathbf{k};\tau) u'_{0\beta}(\mathbf{k};\tau)\rangle}{\delta(\mathbf{k}+\mathbf{k}')} = D_{\alpha\beta}(\mathbf{k}) Q_0(k;\tau,\tau') + \frac{i}{2}\frac{k_a}{k^2}\epsilon_{\alpha\beta a} H_0(k;\tau,\tau')$$

Calculation of the Reynolds stress

$$\left\langle {u'}^\alpha {u'}^\beta \right\rangle = \left\langle {u'_{\mathrm{B}}}^\alpha {u'_{\mathrm{B}}}^\beta \right\rangle + \left\langle {u'_{\mathrm{B}}}^\alpha {u'_{01}}^\beta \right\rangle + \left\langle {u'_{01}}^\alpha {u'_{\mathrm{B}}}^\beta \right\rangle + \cdots$$
$$+ \left\langle {u'_{\mathrm{B}}}^\alpha {u'_{10}}^\beta \right\rangle + \left\langle {u'_{10}}^\alpha {u'_{\mathrm{B}}}^\beta \right\rangle + \cdots$$

$$\left\langle {u'}^\alpha {u'}^\beta \right\rangle_{\mathrm{D}} = -\nu_{\mathrm{T}} \mathcal{S}^{\alpha\beta} + \left[\Gamma^\alpha \left(\Omega^\beta + 2\omega_{\mathrm{F}}^\beta\right) + \Gamma^\beta \left(\Omega^\alpha + 2\omega_{\mathrm{F}}^\alpha\right)\right]_{\mathrm{D}}$$

where $$\mathcal{S}^{\alpha\beta} = \frac{\partial U^\alpha}{\partial x^\beta} + \frac{\partial U^\beta}{\partial x^\alpha} - \frac{2}{3}\nabla\cdot\mathbf{U}\delta^{\alpha\beta}$$

mixing length

$$\nu_{\mathrm{T}} \sim \tau u^2 \sim u\ell$$

Eddy viscosity $$\nu_{\mathrm{T}} = \frac{7}{15}\int d\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k;\tau,\tau_1) Q(k;\tau,\tau_1)$$

Helicity-related coefficient $$\mathbf{\Gamma} = \frac{1}{30}\int k^{-2} d\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k;\tau,\tau_1) \nabla H(k;\tau,\tau_1)$$

helicity inhomogeneity is essential

Eddy viscosity + Helicity model

Reynolds stress

Yokoi & Yoshizawa (1993) Phys. Fluids A**5**, 464

$$\mathcal{R}_{\alpha\beta} \equiv \langle u'_\alpha u'_\beta \rangle$$

$$= \frac{2}{3} K \delta_{\alpha\beta} - \nu_{\rm T} \left(\frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha} \right) + \eta \left[\Omega_\alpha \frac{\partial H}{\partial x_\beta} + \Omega_\beta \frac{\partial H}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \left(\boldsymbol{\Omega} \cdot \nabla \right) H \right]$$

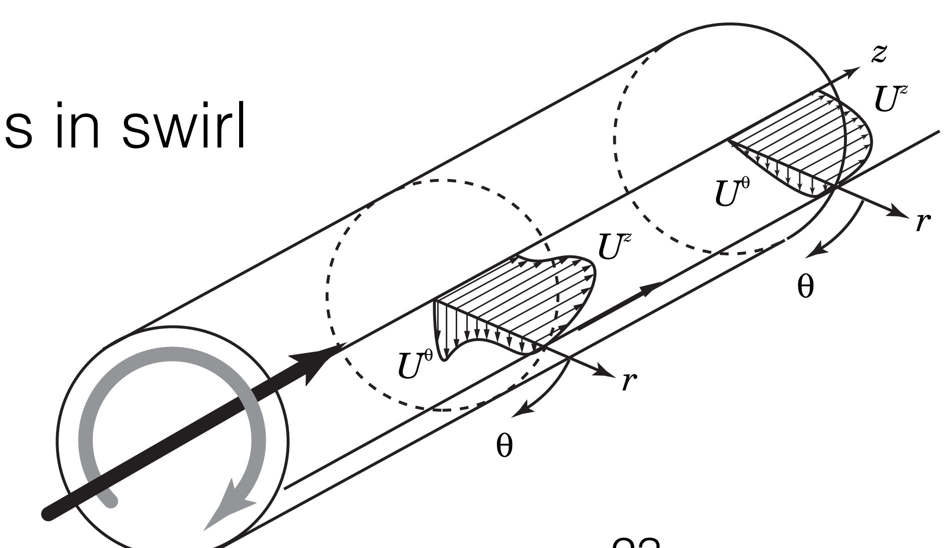
$$\nu_{\rm T} = C_\nu \tau K, \quad \tau = K/\epsilon, \quad \eta = C_H \tau (K^3/\epsilon^2)$$

Turbulence quantities

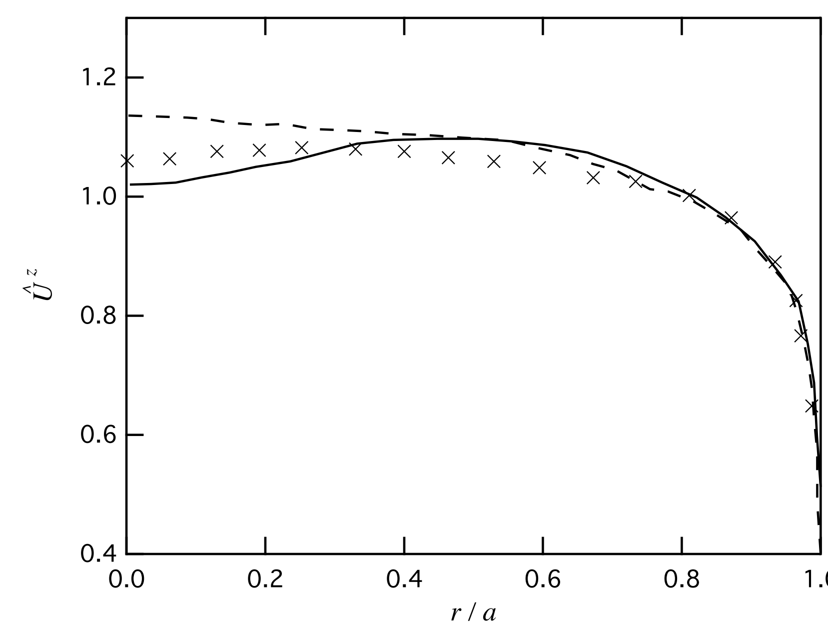
$$K \equiv \frac{1}{2} \langle \mathbf{u}'^2 \rangle, \quad \epsilon \equiv \nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial u'_b}{\partial x_a} \right\rangle,$$

$$H \equiv \langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle, \quad \epsilon_H \equiv 2\nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial \omega'_b}{\partial x_a} \right\rangle$$

Velocity profiles in swirl



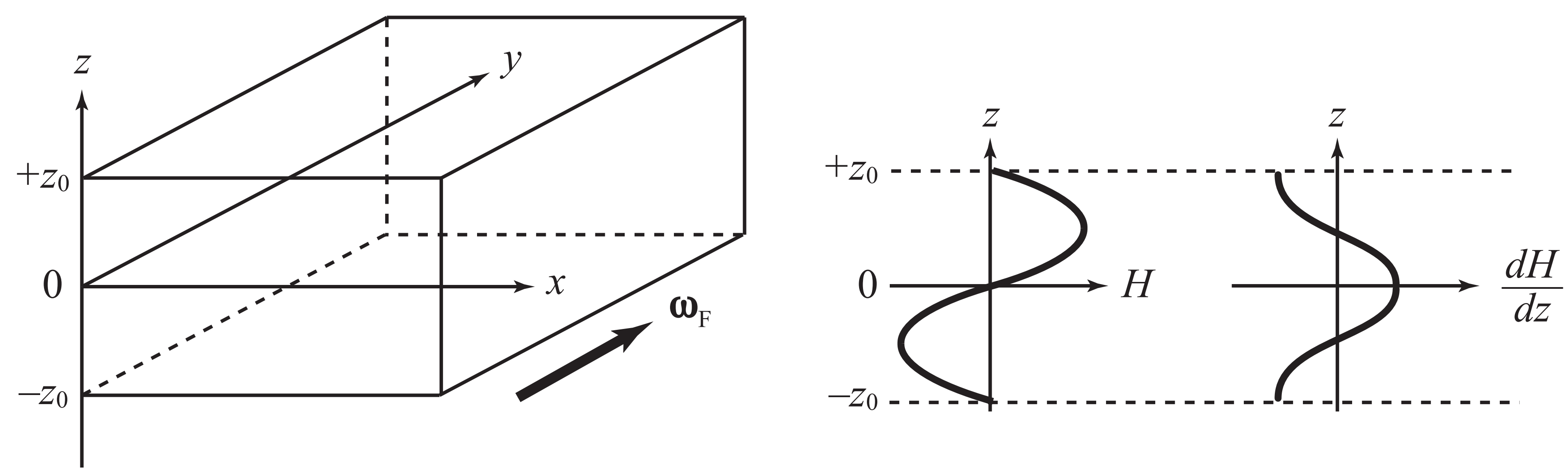
Helicity turbulence model



Yokoi & Brandenburg (2016) Phys. Rev. E. **93**, 033125

DNS set-up

Rotation + Inhomogeneous Helicity (by forcing)



Set-up of the turbulence and rotation $\boldsymbol{\omega}_\mathrm{F}$ (left), the schematic spatial profile of the turbulent helicity *H* (= $\langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$) (center) and its derivative *dH/dz* (right).

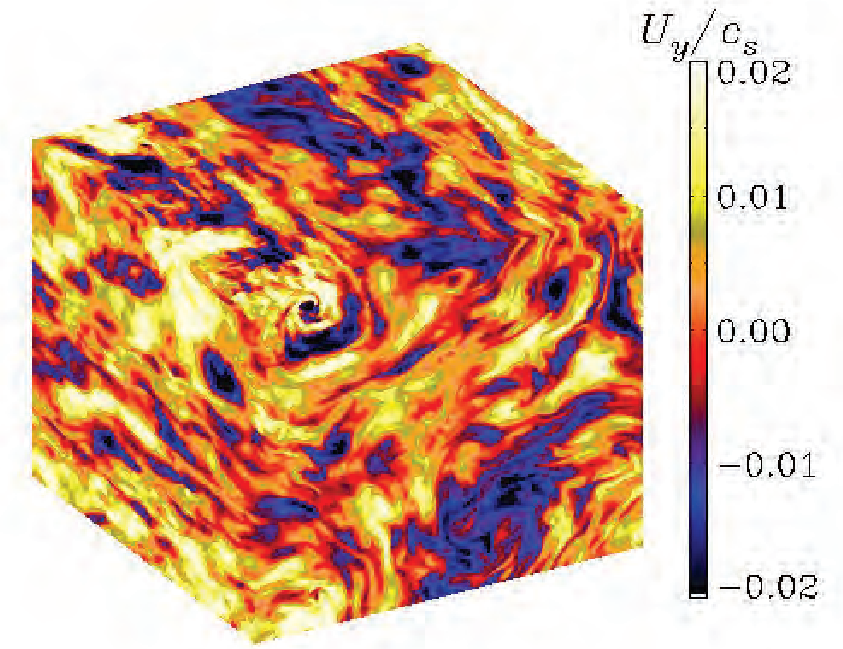
Rotation $\boldsymbol{\omega}_\mathrm{F} = (\omega_\mathrm{F}^x, \omega_\mathrm{F}^y, \omega_\mathrm{F}^z) = (0, \omega_\mathrm{F}, 0)$

Inhomogeneous turbulent helicity $H(z) = H_0 \sin(\pi z / z_0)$

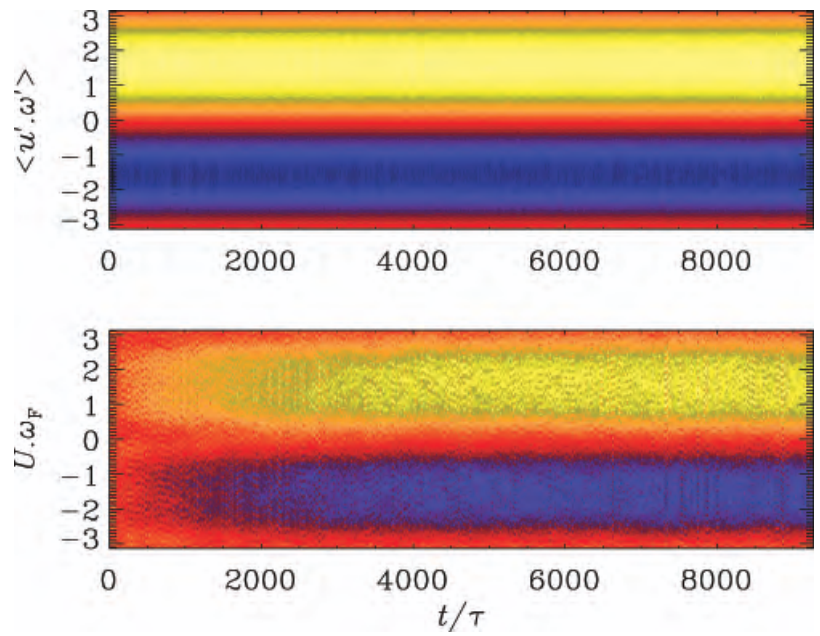
Run	k_f/k_1	Re	Co	$\eta/(\nu_\mathrm{T}\tau^2)$
A	15	60	0.74	0.22
B1	5	150	2.6	0.27
B2	5	460	1.7	0.27
B3	5	980	1.6	0.51
C1	30	18	0.63	0.50
C2	30	80	0.55	0.03
C3	30	100	0.46	0.08

Summary of DNS results

Global flow generation



Axial flow component U_y on the periphery of the domain



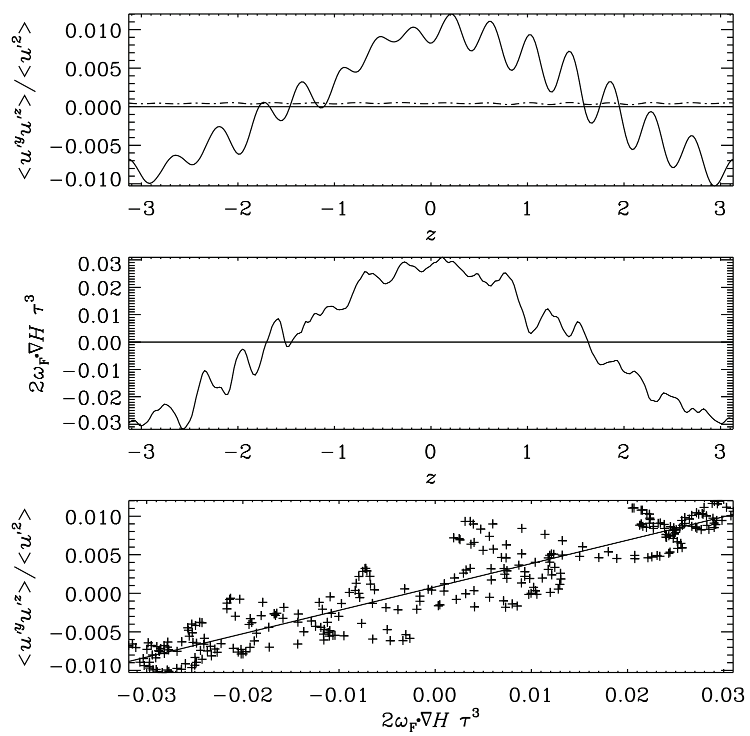
Turbulent helicity $\langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$ (top) and mean-flow helicity $\mathbf{U} \cdot 2\boldsymbol{\omega}_F$ (bottom)

Reynolds stress

$$\left\langle u'^{\alpha} u'^{\beta} \right\rangle_{\mathrm{D}} = -\nu_{\mathrm{T}} \mathcal{S}^{\alpha\beta} + \left[\Gamma^{\alpha} \left(\Omega^{\beta} + 2\omega_{\mathrm{F}}^{\beta} \right) + \Gamma^{\beta} \left(\Omega^{\alpha} + 2\omega_{\mathrm{F}}^{\alpha} \right) \right]_{\mathrm{D}}$$

Early stage

$$\left\langle u'^{y} u'^{z} \right\rangle = \eta 2\omega_{\mathrm{F}}^{y} \frac{\partial H}{\partial z}$$



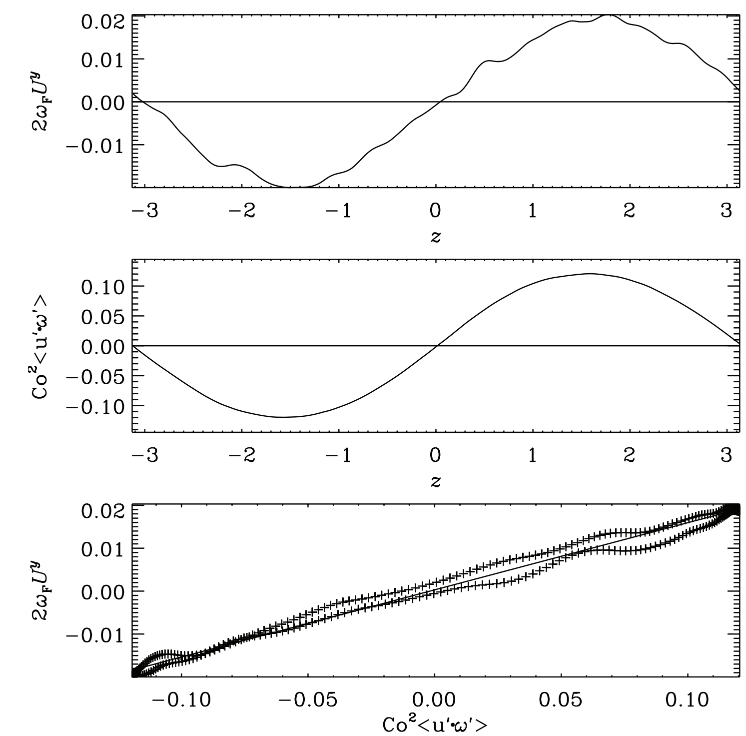
Reynolds stress $\langle u'^{y} u'^{z} \rangle$ (top),

helicity-effect term $(\nabla H)^{z}\, 2\omega_{\mathrm{F}}^{y}$ (middle), and their correlation (bottom).

Developed stage

$$\left\langle u'^{y} u'^{z} \right\rangle = -\nu_{\mathrm{T}} \frac{\partial U^{y}}{\partial z} + \eta 2\omega_{\mathrm{F}}^{y} \frac{\partial H}{\partial z}$$

$$U^{y} = (\eta/\nu_{\mathrm{T}})\, 2\omega_{\mathrm{F}}^{y} H$$



Mean axial velocity U^{y} (top), turbulent helicity multiplied by rotation $2\omega_{\mathrm{F}}H$ (middle), and their correlation (bottom).

Physical origin

Reynolds stress $\mathcal{R}^{ij} \equiv \langle u'^i u'^j \rangle$

Vortexmotive force $\mathbf{V}_\mathrm{M} \equiv \langle \mathbf{u}' \times \boldsymbol{\omega}' \rangle$

$$V^i_\mathrm{M} = -\frac{\partial \mathcal{R}^{ij}}{\partial x^j} + \frac{\partial K}{\partial x^i}$$

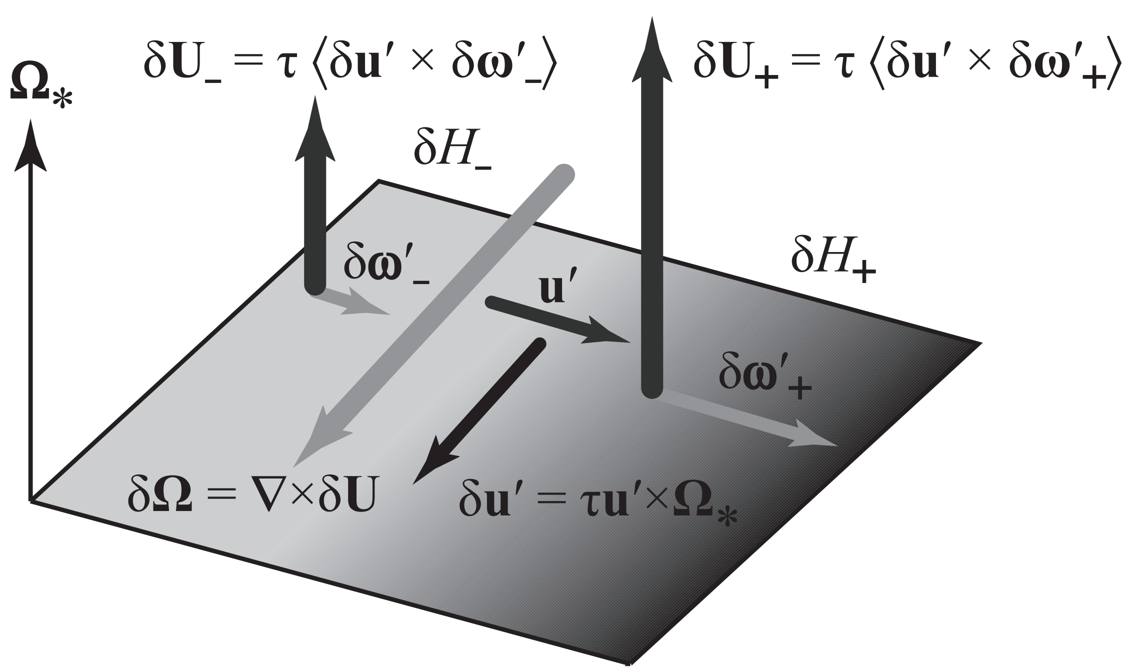
$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\mathbf{U} \times \boldsymbol{\Omega}) + \nabla \times \mathbf{V}_\mathrm{M} + \nu \nabla^2 \boldsymbol{\Omega}$$

$$\mathbf{V}_\mathrm{M} = -D_\Gamma 2\boldsymbol{\omega}_\mathrm{F} - \nu_\mathrm{T} \nabla \times \boldsymbol{\Omega}$$

$$D_\Gamma = \nabla \cdot \boldsymbol{\Gamma} \propto \nabla^2 H$$

→ $$\delta \mathbf{U} \sim -(\nabla^2 H) \boldsymbol{\Omega}_*$$

$$\nabla^2 H \simeq -\frac{\delta H}{\ell^2} = -\frac{\langle \mathbf{u}' \cdot \delta \boldsymbol{\omega}' \rangle}{\ell^2}$$



Reynolds-stress budget

Inagaki, Yokoi & Hamba, Phys. Rev. Fluids, **2**, 114605 (2017)

Local helical forcing

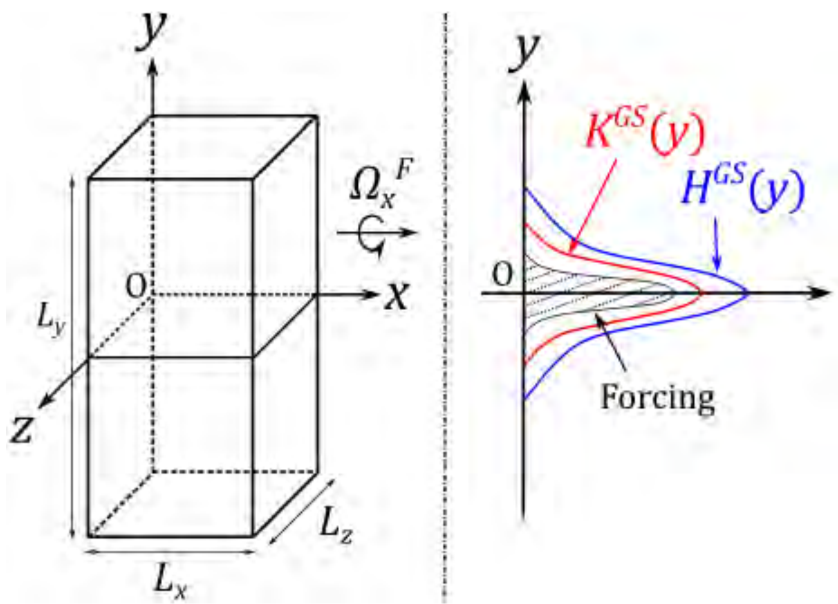
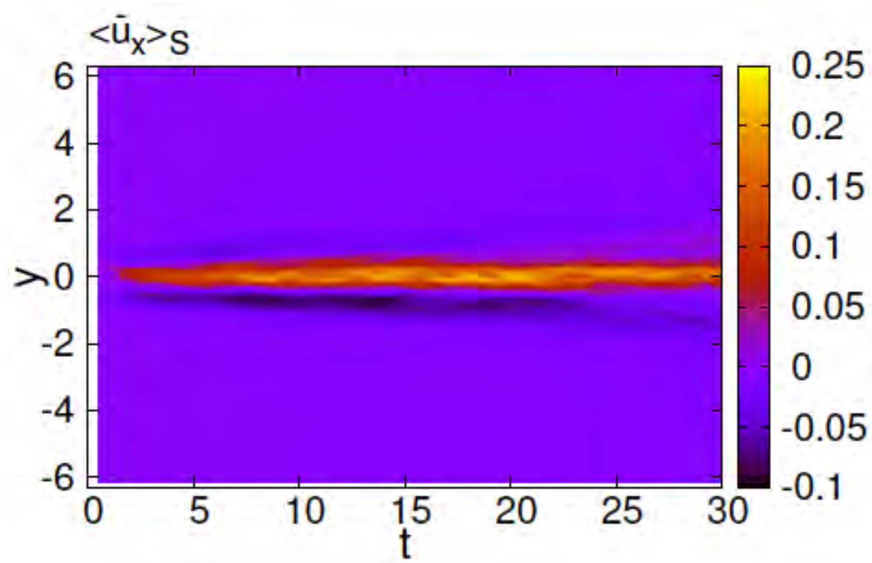
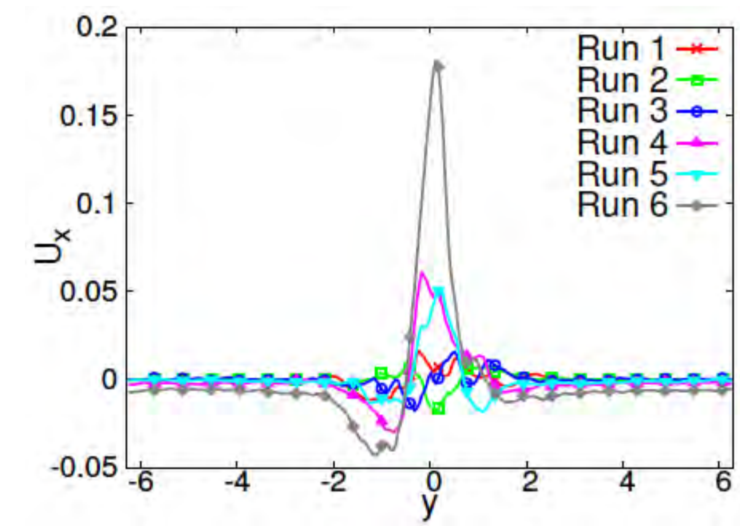


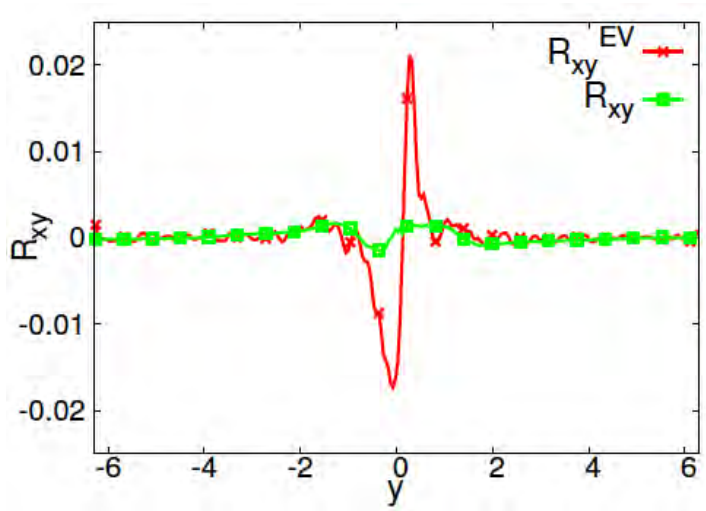
TABLE I. Calculation parameters.

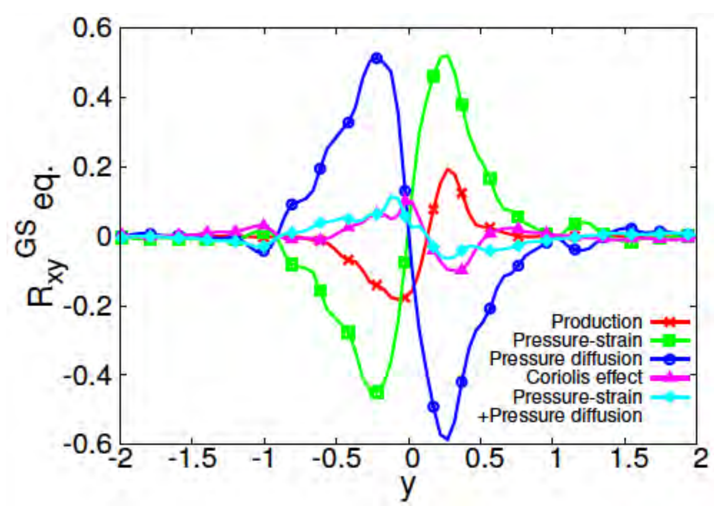
Run	α	Ω_x^F	L_0^{GS}	Ro_0^{GS}
1	0	0	0.506	∞
2	0.5	0	0.547	∞
3	0	5	0.542	0.185
4	0.2	5	0.550	0.182
5	0.5	2	0.544	0.459
6	0.5	5	0.602	0.166





$$R_{xy} = \nu_{\mathrm{T}} \frac{\partial U_x}{\partial y} + N_{xy}$$





$$\frac{\partial R_{xy}^{GS}}{\partial t} \simeq P_{xy}^{GS} + \Phi_{xy}^{GS} + \Pi_{xy}^{GS} + C_{xy}^{GS} \simeq 0$$

Press. diff. $\Pi_{xy}^{GS} = -\frac{\partial}{\partial y} \langle \overline{p}' \overline{u}'_x \rangle$

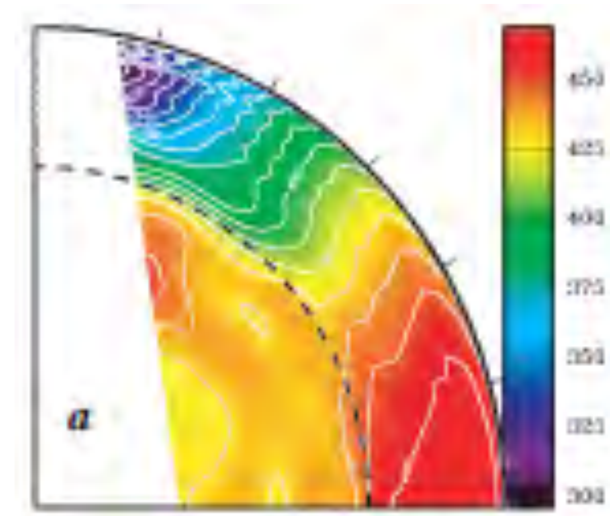
Coriolis $C_{xy}^{GS} = 2 R_{xz}^{GS} \Omega_x^F$

Angular-momentum transport in the solar convection zone

Angular momentum around the rotation axis

$$L = \Gamma r^2 \omega_{\rm F} + \Gamma r U^\phi \qquad \Gamma = \sin\theta$$

$$\frac{\partial}{\partial t}\rho L + \nabla \cdot (\rho \mathbf{F}_L) = 0$$



Miesch (2005) Liv. Rev. Sol. Phys. 2005-1

Vector flux of angular momentum $\mathbf{F}_L$

$$F_L^r = L U^r + r\Gamma \mathcal{R}^{r\phi}$$

$$F_L^\theta = L U^\theta + r\Gamma \mathcal{R}^{\theta\phi}$$

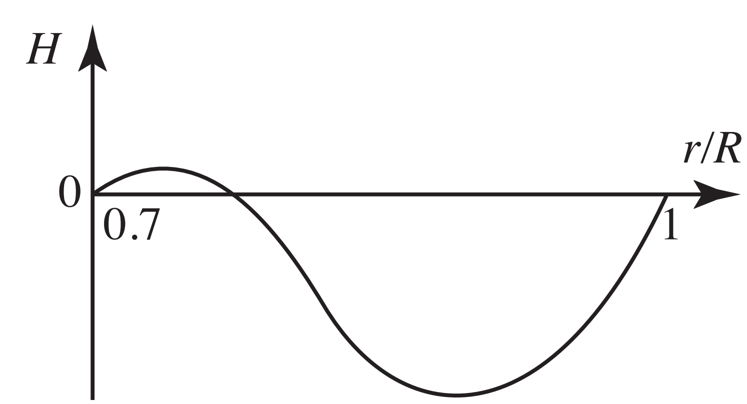
$$\delta \mathbf{U} \sim -(\nabla^2 H)\mathbf{\Omega}_*$$

Helicity effect

$$\mathcal{R}_H^{r\phi} = +\frac{\partial H}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r} r U^\theta - \frac{1}{r}\frac{\partial U^r}{\partial \theta}\right)$$

$$\mathcal{R}_H^{\theta\phi} = +\frac{1}{r}\frac{\partial H}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r} r U^\theta - \frac{1}{r}\frac{\partial U^r}{\partial \theta}\right)$$

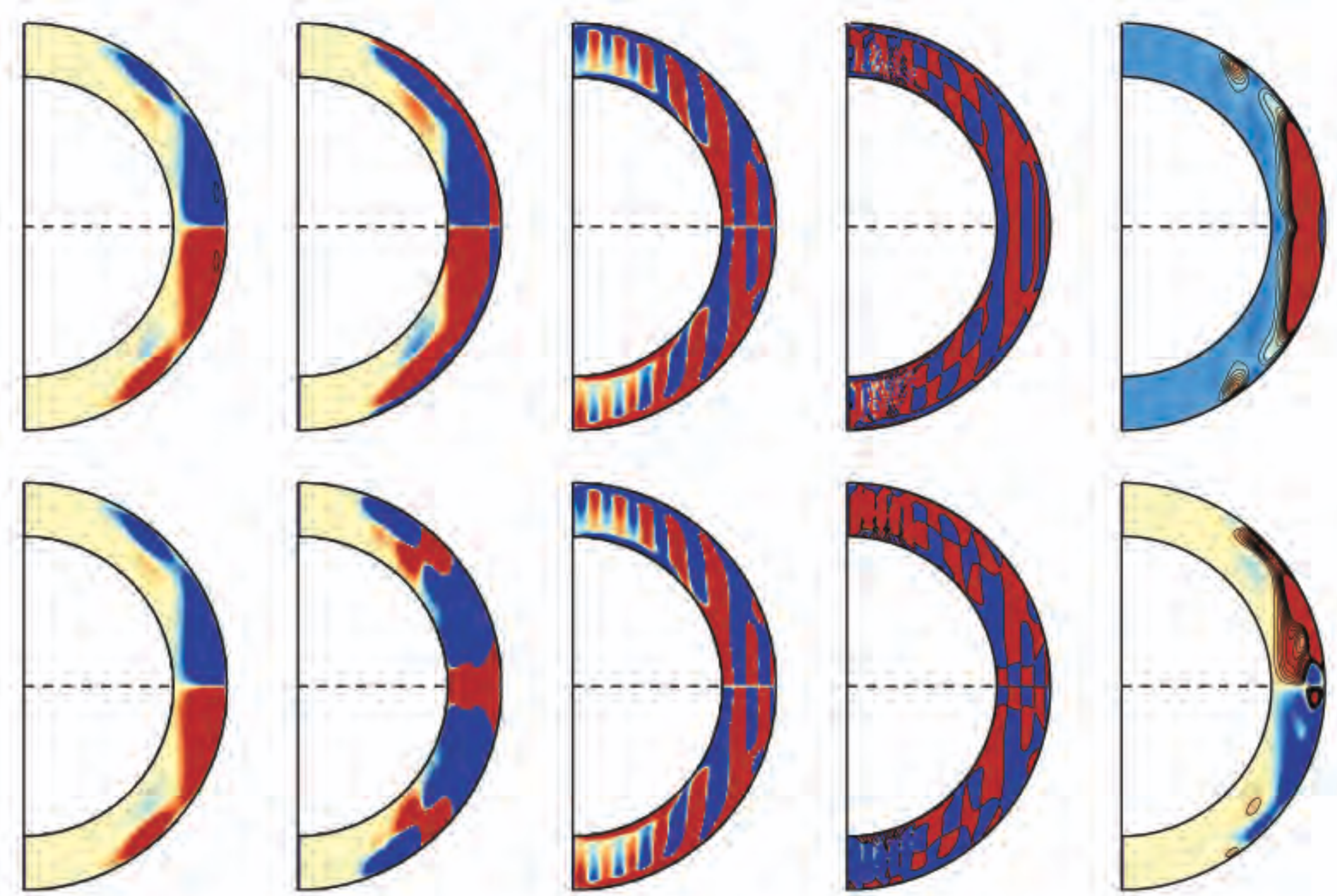
Schematic helicity distribution



$$\nabla^2 H > 0$$

Helicity effect in the Reynolds stress

Helicity	Helicity Gradient	Azimuthal Vorticity	Helicity effect	Reynolds stress
$\mathbf{u}\cdot\boldsymbol{\omega}$	$\dfrac{\partial H}{\partial r}$	$\overline{\Omega}^{\phi}$	$\dfrac{\partial H}{\partial r}\overline{\Omega}^{\phi}$	$\overline{u'^r u'^\phi}$



$\mathbf{u}\cdot\boldsymbol{\omega} - \overline{\mathbf{u}}\cdot\overline{\boldsymbol{\omega}}$ $(\equiv H)$	$\dfrac{1}{r}\dfrac{\partial H}{\partial \theta}$	$\overline{\Omega}^{\phi}$	$\dfrac{1}{r}\dfrac{\partial H}{\partial \theta}\overline{\Omega}^{\phi}$	$\overline{u'^\theta u'^\phi}$

(provided by Mark Miesch)

$C_\eta \tau \ell^2 |(\nabla^2 H)\Omega_*|$

Solar parameters

$$v \sim 200\ \mathrm{m\ s^{-1}} = 2\times 10^4 \mathrm{cm\ s^{-1}}$$
$$\ell \sim 200\ \mathrm{Mm} = 2\times 10^{10}\mathrm{cm}$$
$$\tau \sim \ell/v \sim 10^6\ \mathrm{s}$$

$r\phi$ component

$$\left|\overline{u'^r u'^\phi}\right| \sim 1.2\times 10^9$$
$$\left|\frac{\partial H}{\partial r}\overline{\Omega}^\phi\right| \sim 9.4\times 10^{-15}$$
$$\tau\ell^2 \left|\frac{\partial H}{\partial r}\overline{\Omega}^\phi\right| \sim 10^{12} \rightarrow 10^9$$

with $C_\eta = O(10^{-3})$

$\theta\phi$ component

$$\left|\overline{u'^\theta u'^\phi}\right| \sim 5.6\times 10^8$$
$$\left|\frac{1}{r}\frac{\partial H}{\partial \theta}\overline{\Omega}^\phi\right| \sim 2.6\times 10^{-15}$$
$$\tau\ell^2 \left|\frac{1}{r}\frac{\partial H}{\partial \theta}\overline{\Omega}^\phi\right| \sim 10^{11} \rightarrow 10^8$$

Magnitude same as the Reynolds stress

Large-scale flow generation by cross helicity

Reynolds and turbulent Maxwell stress

eddy viscosity — inhomogeneous helicity

$$\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle_{\mathrm{D}} = -\nu_{\mathrm{K}} \boldsymbol{\mathcal{S}} + \nu_{\mathrm{M}} \boldsymbol{\mathcal{M}} + \eta_H \boldsymbol{\Omega}_* \nabla H + \cdots$$

cross helicity

D: deviatoric part

$\boldsymbol{\mathcal{S}}$: mean velocity strain

$\boldsymbol{\mathcal{M}}$: mean magnetic-field strain

$\boldsymbol{\Omega}_*$: absolute mean vorticity (mean vorticity + rotation)

cf. $\langle \mathbf{u}' \times \mathbf{b}' \rangle = -\eta_{\mathrm{T}} \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U} + \alpha \mathbf{B} + \cdots$

Turbulent cross helicity coupled with **mean magnetic-field strain** may contribute to **transport suppression** and/or **global flow generation** against the eddy-viscosity effect

Physical interpretation of large-scale flow generation by cross helicity

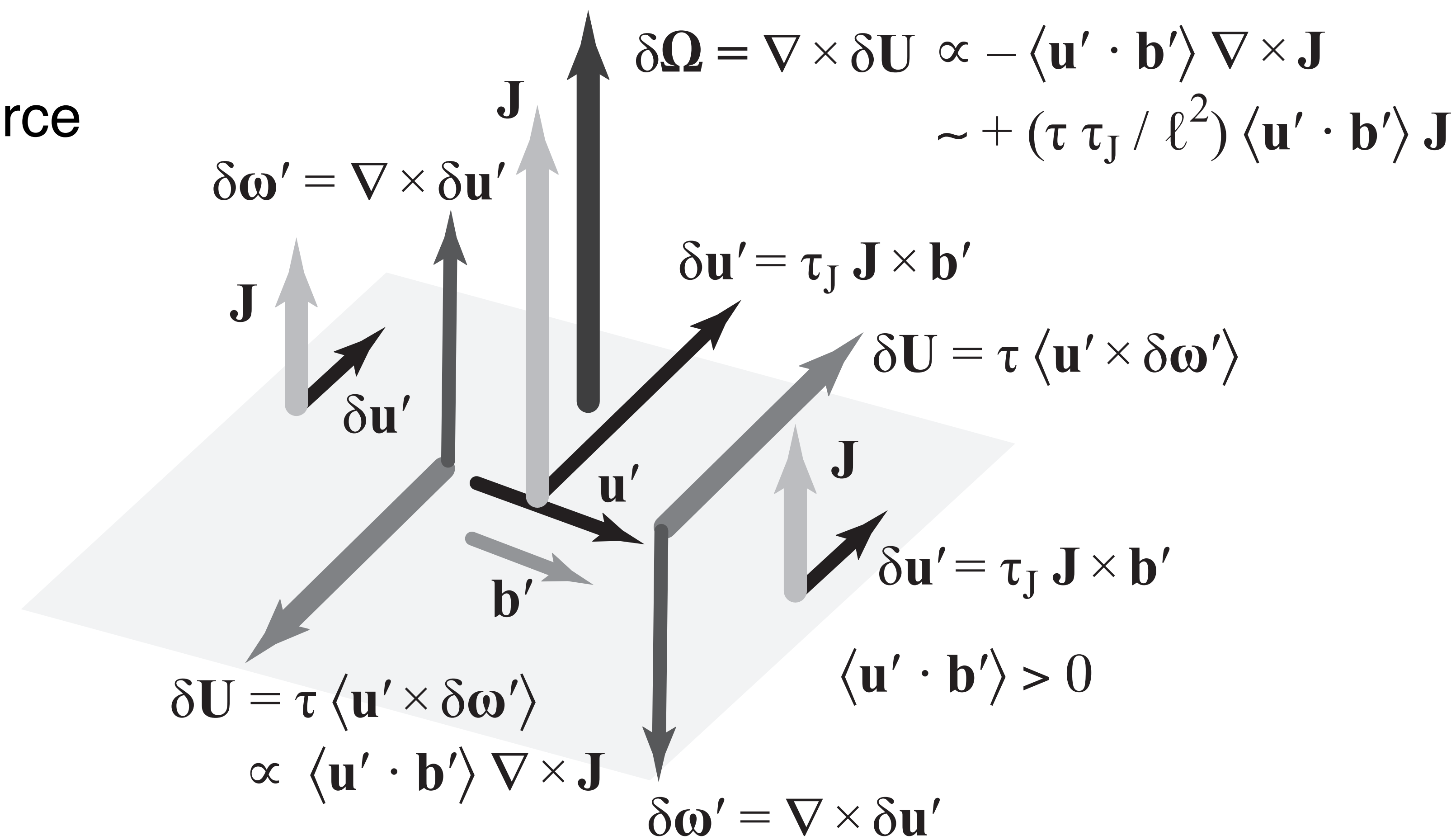
Velocity fluctuation
induced by **fluctuating Lorentz force**

$$\delta \mathbf{u}' = \tau_J \mathbf{J} \times \mathbf{b}'$$

Associated vorticity

$$\delta \boldsymbol{\omega}' = \nabla \times \delta \mathbf{u}'$$
$$= \tau_J \nabla \times (\mathbf{J} \times \mathbf{b}')$$
$$\simeq \tau_J (\mathbf{b}' \cdot \nabla) \mathbf{J}$$

Mean electric-current distribution



Large-scale flow induction due to cross helicity

$$\delta \mathbf{U} = \tau \langle \mathbf{u}' \times \delta \boldsymbol{\omega}' \rangle \propto \langle \mathbf{u}' \cdot \mathbf{b}' \rangle \nabla \times \mathbf{J}$$ in the direction of $\nabla \times \mathbf{J}$

$$\delta \boldsymbol{\Omega} = \nabla \times \mathbf{U} = -\tau \tau_J \langle \mathbf{u}' \cdot \mathbf{b}' \rangle \nabla^2 \mathbf{J} \simeq + \frac{\tau \tau_J}{\ell^2} \langle \mathbf{u}' \cdot \mathbf{b}' \rangle \mathbf{J}$$ in the direction of $\mathbf{J}$

Summary

- Formulation for strongly non-linear and inhomogeneous/anisotropic turbulence
- Dynamo (or transport suppression) by cross helicity
- Flow generation (or momentum transport suppression) by kinetic helicity and cross helicity