INVERSE CASCADE OF ENERGY
IN HELICAL TURBULENCE

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Inverse cascade of energy

2D turbulence ☑

Two inviscid quadratic invariants:

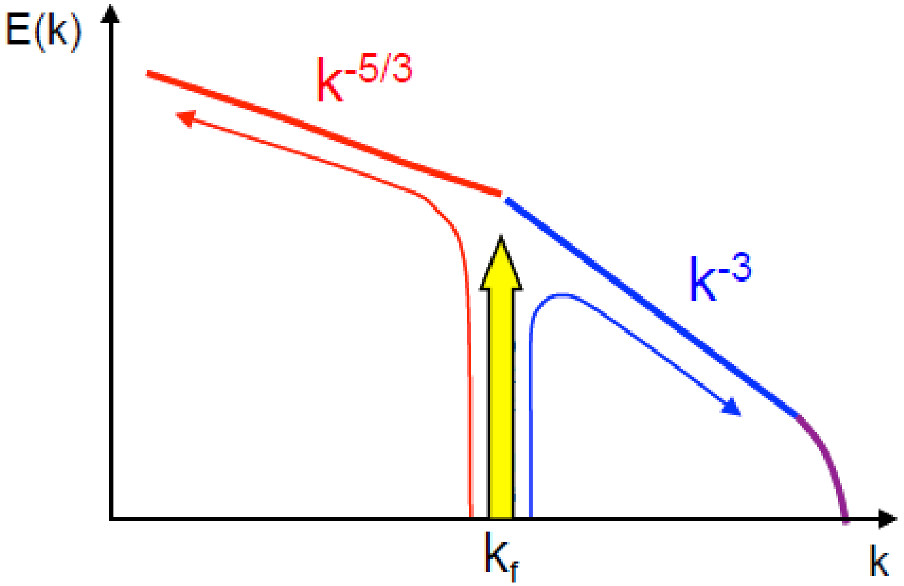
Energy $E = \frac{1}{2} < |\boldsymbol{u}|^2 >,$

Enstrophy $Z = \frac{1}{2} < |\boldsymbol{\omega}|^2 >, \quad \boldsymbol{\omega} = \nabla \times \boldsymbol{u}$

Both are definite positive

$\Rightarrow$ - Inverse cascade of energy
- Direct cascade of enstrophy

(Fjørtoft 1953, Kraichnan 1967)



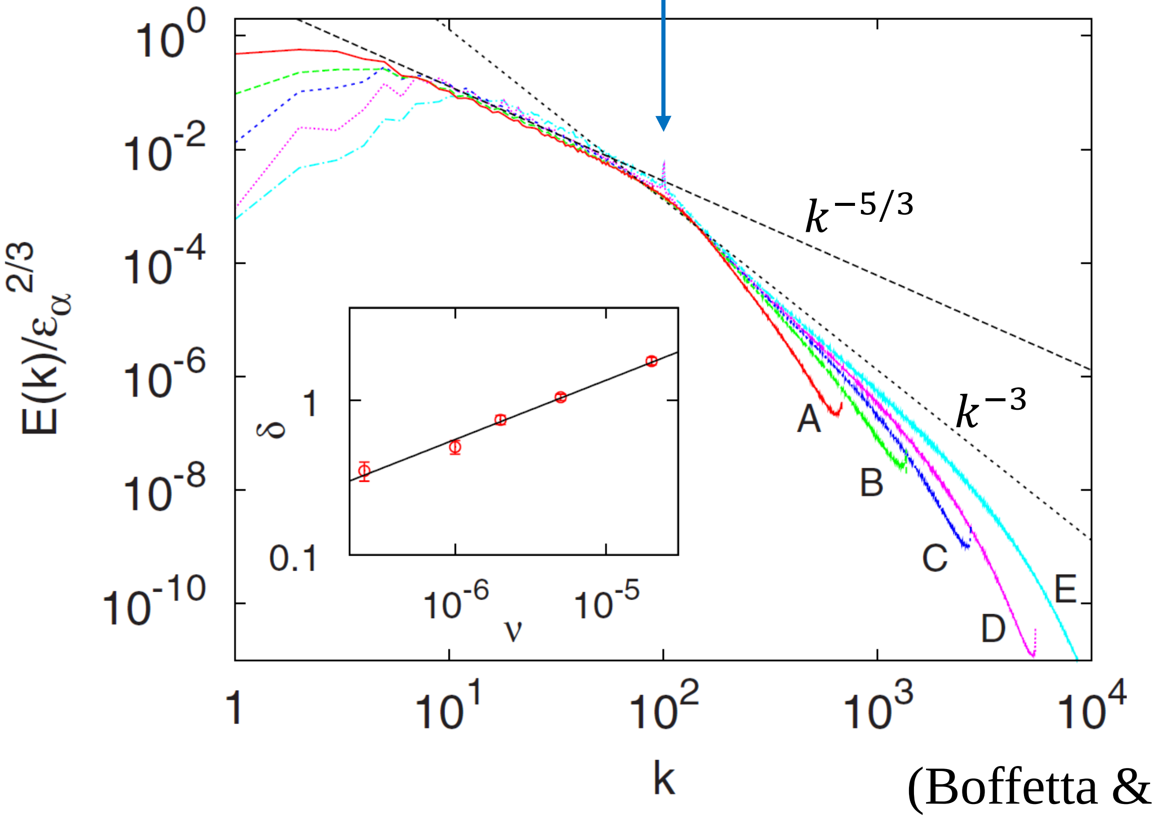
Inverse cascade of energy

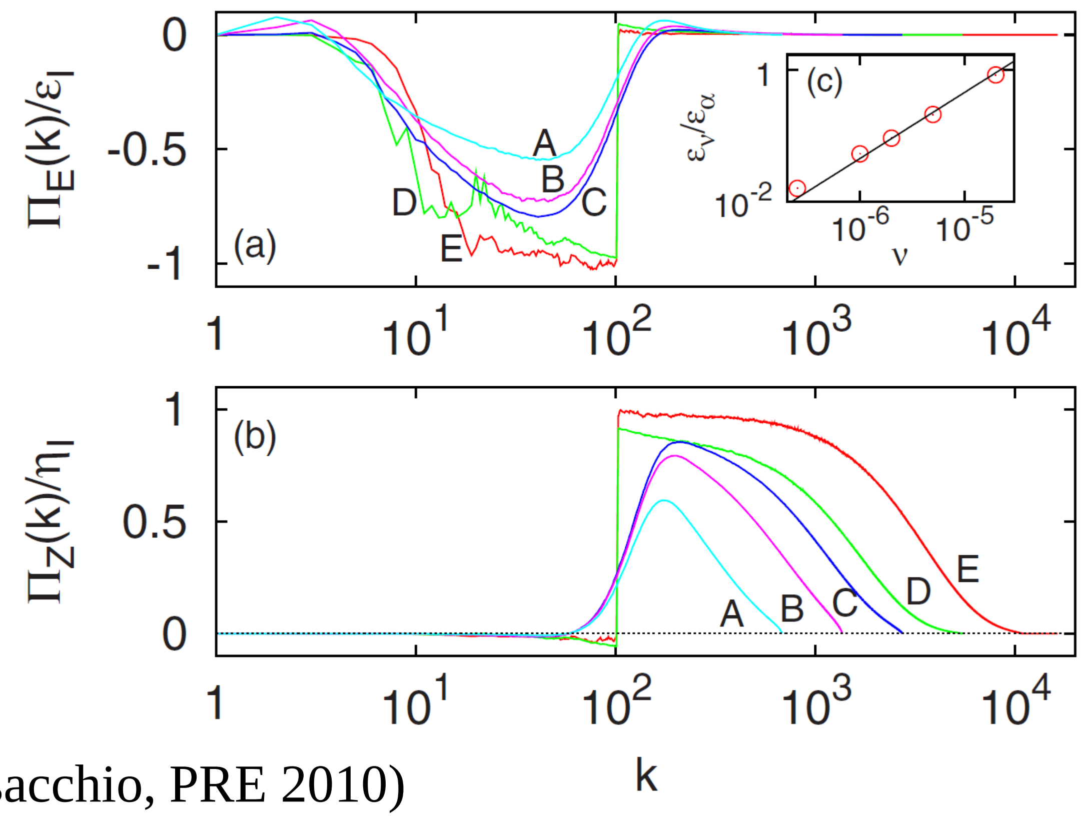
2D turbulence ☑

Energy spectra

Energy / enstrophy fluxes in spectral space

Injection scale of energy



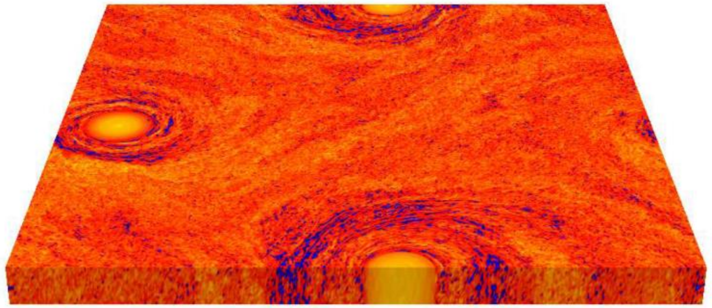


(Boffetta & Musacchio, PRE 2010)

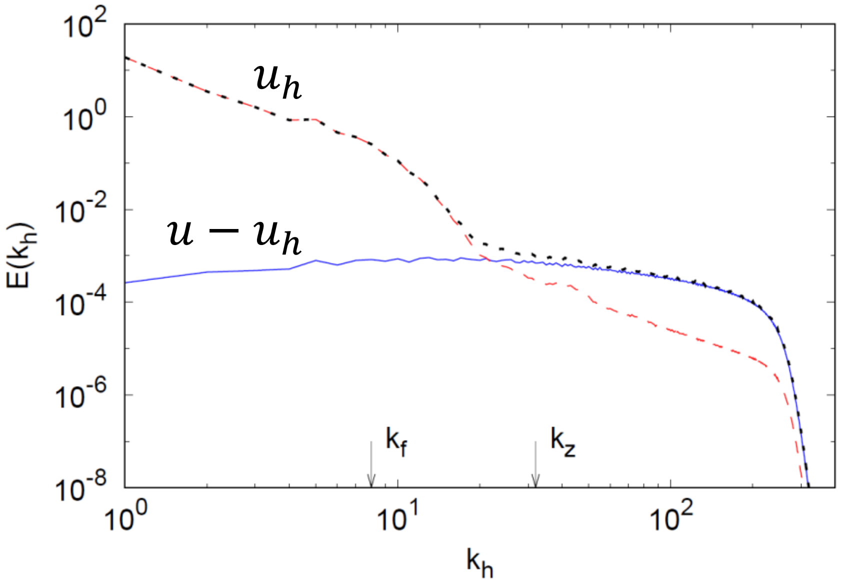
Inverse cascade of energy

2D turbulence ☑

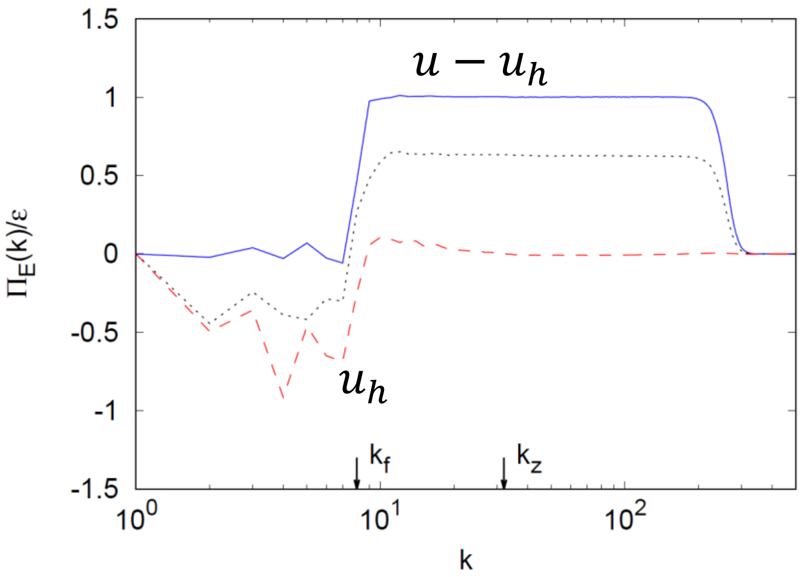
3D turbulence in a slab ☑



Energy spectra



Energy flux



(Musacchio & Boffetta, PRF 2019)

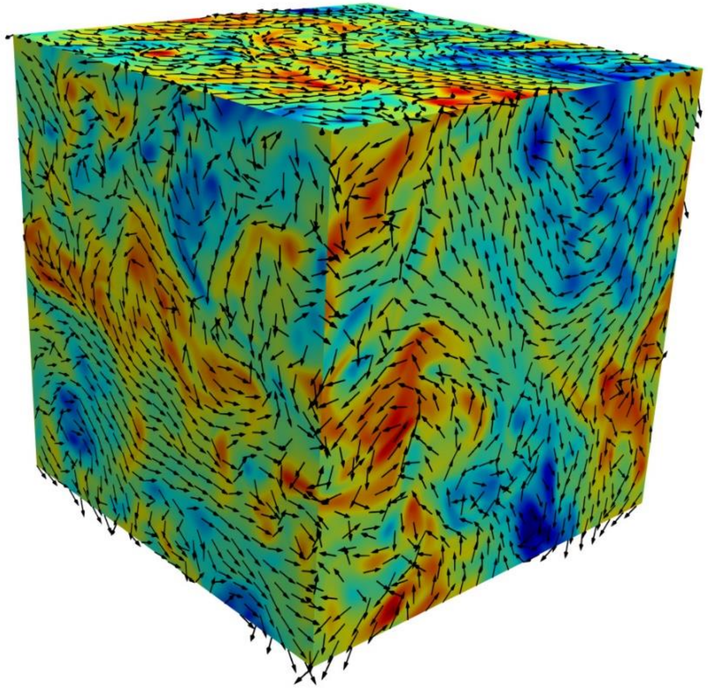
Inverse cascade of energy

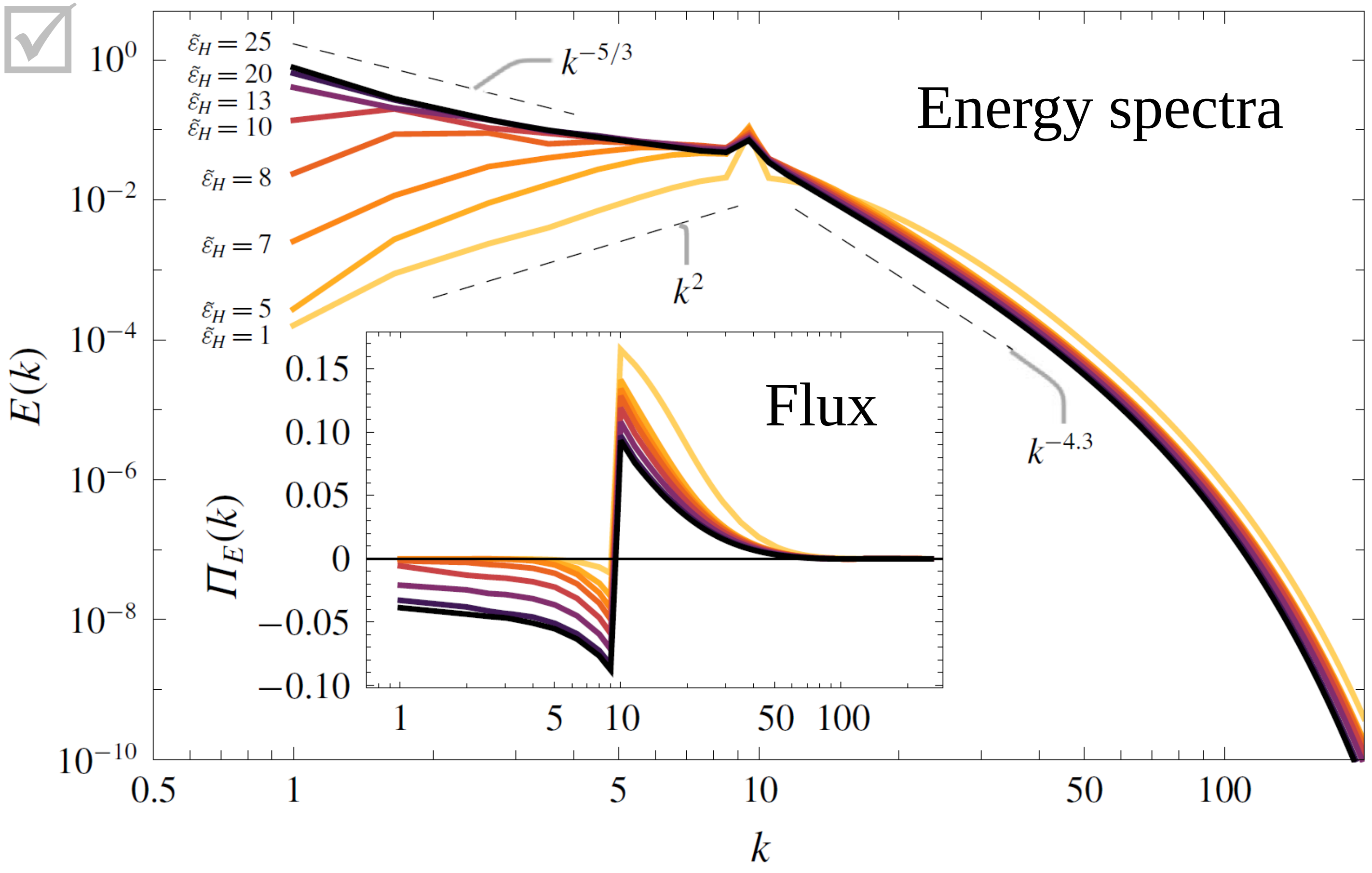
2D turbulence ☑

3D turbulence in a slab ☑

3D helical turbulence ☑

Velocity field





(Plunian et al., JFM 2020)

In 3D turbulence

Two inviscid quadratic invariants

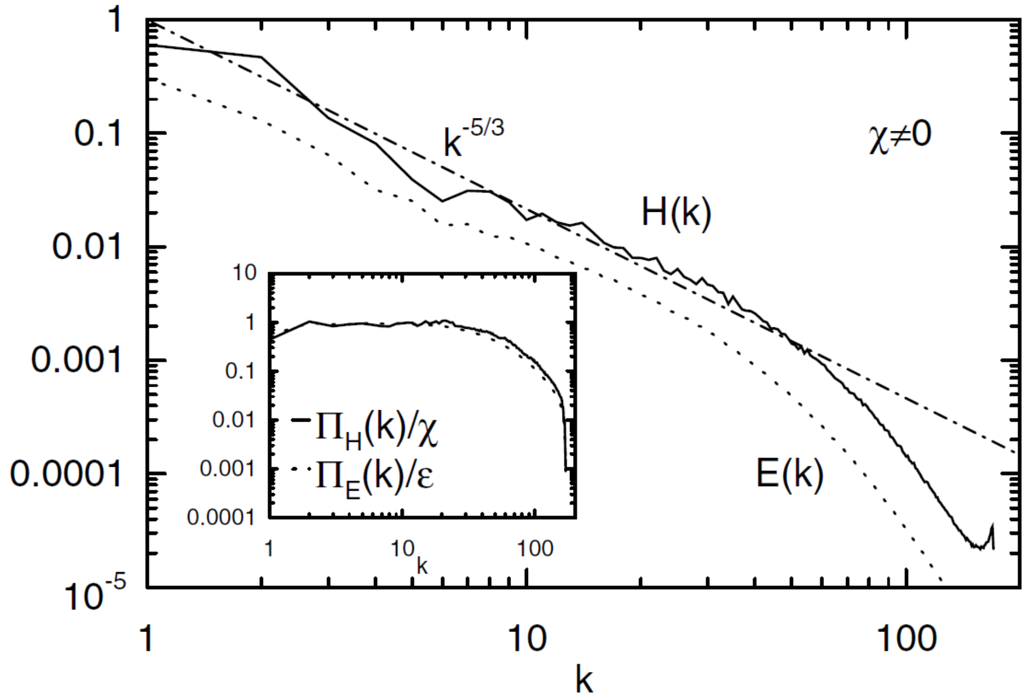
Energy $E = \frac{1}{2} < |\boldsymbol{u}|^2 >,$

Helicity $H = \frac{1}{2} \boldsymbol{u}.\boldsymbol{\omega},$ $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$

Only energy is definite positive

$\Rightarrow$
- Direct cascade of energy
- Direct cascade of helicity

(Chen, Chen, Eyink & Holm, PRL 2003)



Helical mode decomposition

In spectral space

$$\mathbf{u}(\mathbf{k}) = \mathbf{u}^+(\mathbf{k}) + \mathbf{u}^-(\mathbf{k})$$
$$= u^+(\mathbf{k})\mathbf{h}^+(\mathbf{k}) + u^-(\mathbf{k})\mathbf{h}^-(\mathbf{k})$$

where $u^\pm$ are complex scalars and $\mathbf{h}^\pm$ are the eigenvectors of the curl operator satisfying $i\mathbf{k} \times \mathbf{h}^\pm = \pm|\mathbf{k}|\mathbf{h}^\pm$ [20,21].

F. Waleffe, Phys. Fluids **4**, 350 (1992)

$$E^u(k) = \frac{1}{2}\left(|u^+(\boldsymbol{k})|^2 + |u^-(\boldsymbol{k})|^2\right)$$

$$H^u(k) = \frac{k}{2}\left(|u^+(\boldsymbol{k})|^2 - |u^-(\boldsymbol{k})|^2\right)$$

- Non helical state $\Leftrightarrow$ $|u^+(\boldsymbol{k})| = |u^-(\boldsymbol{k})|$
- **Maximal Helical state** $\Leftrightarrow$ $|u^+(\boldsymbol{k})| = 0$, or $|u^-(\boldsymbol{k})| = 0$

In a maximal helical state both energy and helicity are sign definite

2D Turbulence

Maximal Helical State

Two inviscid quadratic invariants

2D Turbulence

Energy $E(k) = \frac{1}{2} < |u^{2D}(\boldsymbol{k})|^2 >$

Enstrophy $Z(k) = \frac{k^2}{2} < |u^{2D}(\boldsymbol{k})|^2 >$

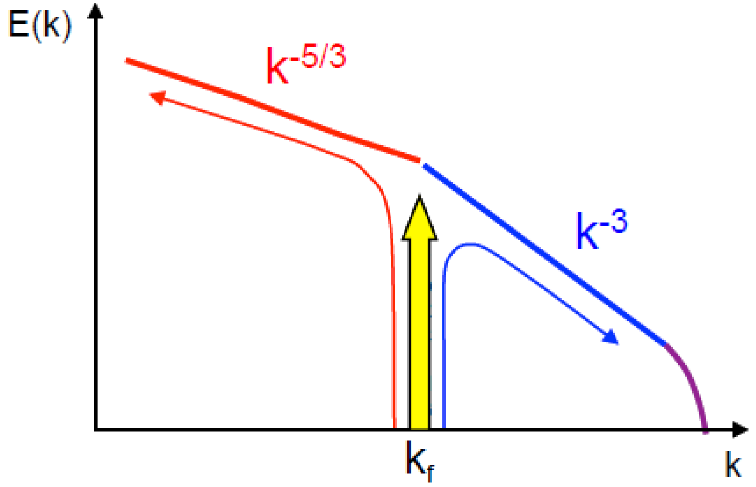
Maximal Helical State

Energy $E(k) = \frac{1}{2} < |u^{+}(\boldsymbol{k})|^2 >$

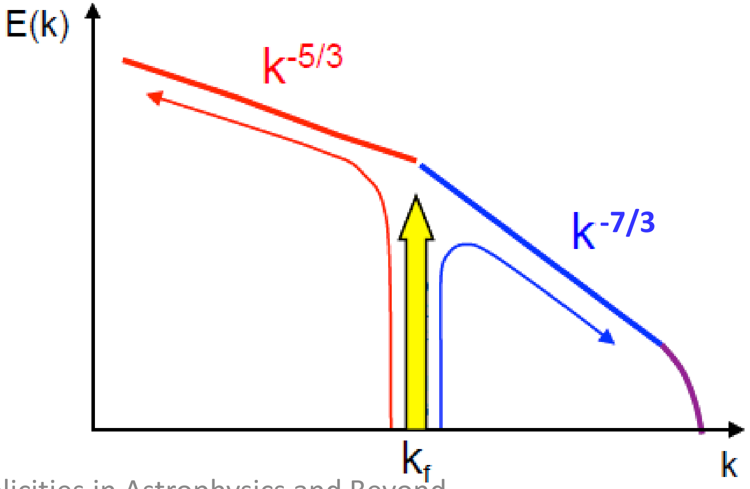
Helicity $H(k) = \pm \frac{k}{2} < \left|u^{\pm}(\boldsymbol{k})\right|^2 >$

Both are sign definite

- Inverse cascade of energy
- Direct cascade of enstrophy



- Inverse cascade of energy
- Direct cascade of helicity

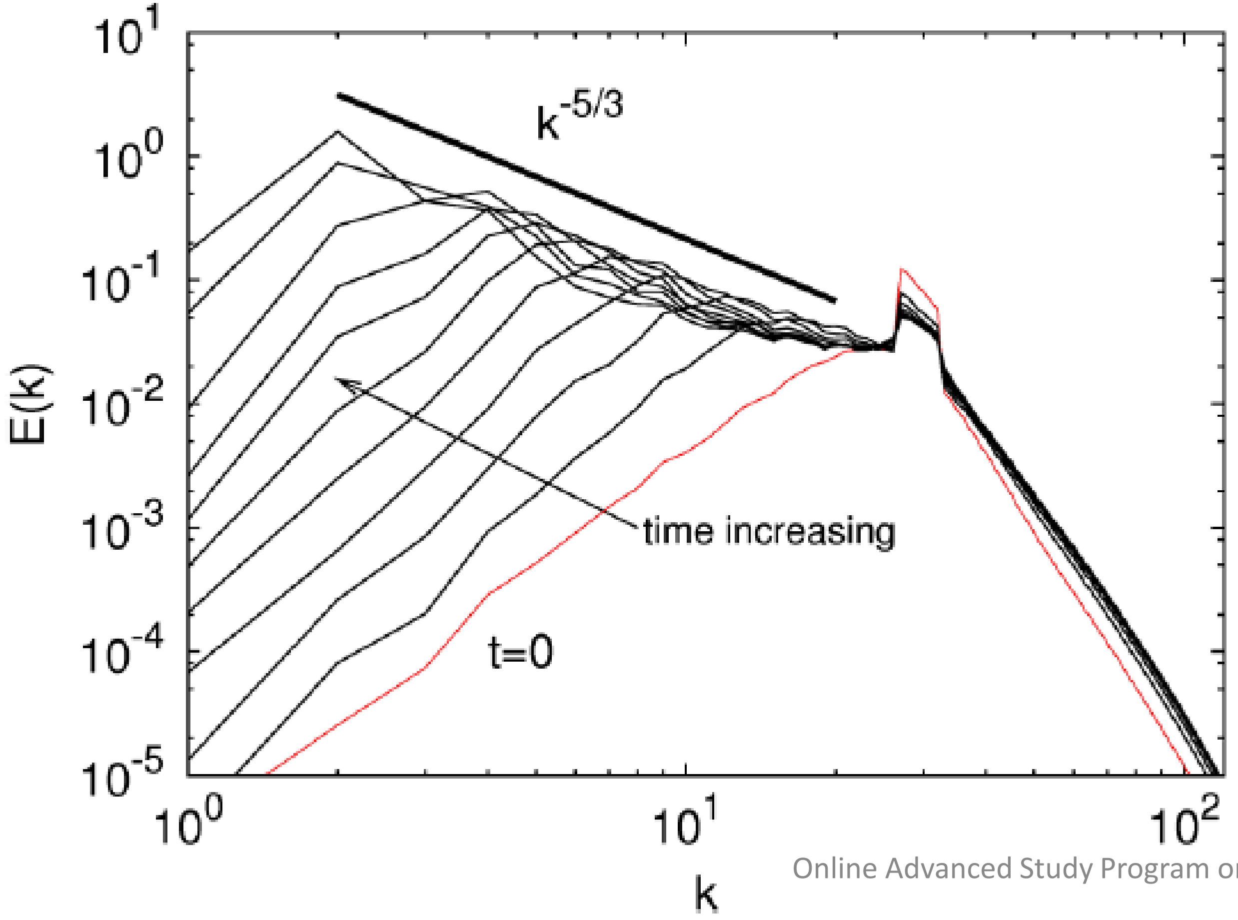


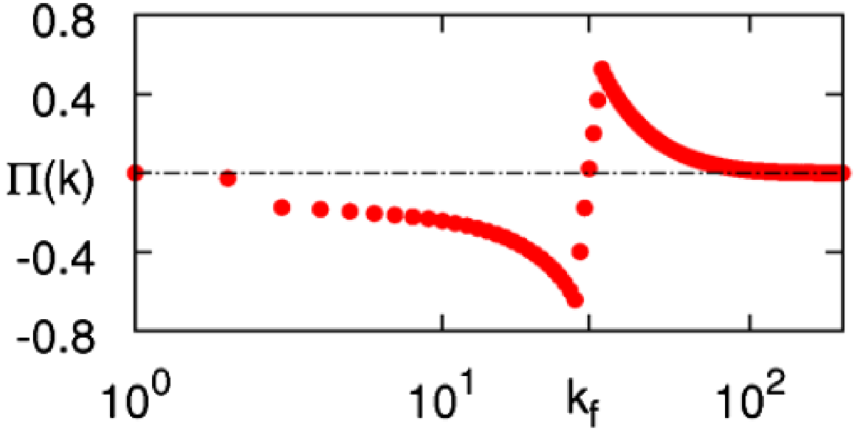
Solving the decimated Navier-Stokes equations

Possible triads

$\boxed{(\boldsymbol{v}^+, \boldsymbol{v}^+, \boldsymbol{v}^+)}$
$(\boldsymbol{v}^+, \boldsymbol{v}^+, \boldsymbol{v}^-)$
$(\boldsymbol{v}^+, \boldsymbol{v}^-, \boldsymbol{v}^-)$
$(\boldsymbol{v}^-, \boldsymbol{v}^-, \boldsymbol{v}^-)$

$$\partial_t \boldsymbol{v}^+ = (-\boldsymbol{v}^+ \cdot \nabla \boldsymbol{v}^+ - \nabla p)^+ + \nu \Delta \boldsymbol{v}^+ + \boldsymbol{f}^+,$$





Biferale et al (2012, 2013)

Solving the full Navier-Stokes equations

$$\partial_t \mathbf{u} = -(\mathbf{u} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \mathbf{u} + \mu \nabla^{-2} \mathbf{u} + \mathbf{f},$$

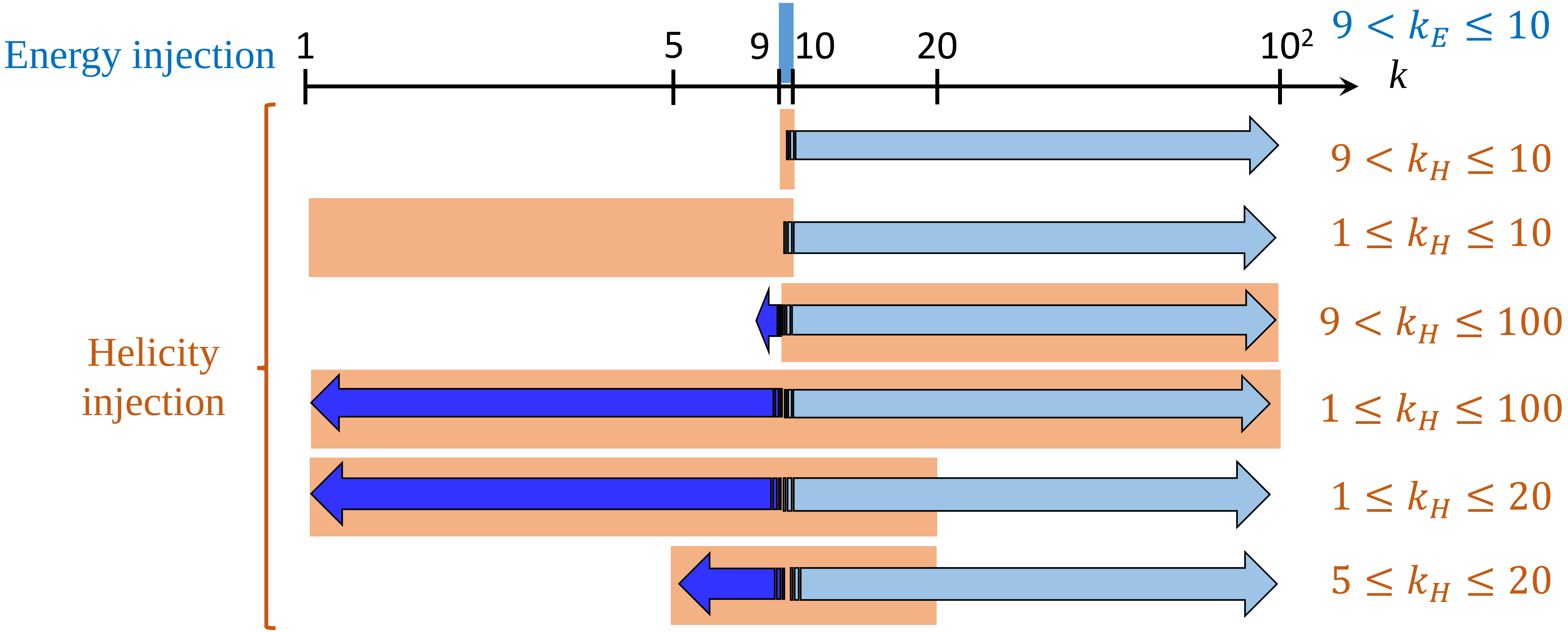
Separated injection scales of energy and helicity

$$\mathbf{f}(\mathbf{k}) = c^{+}(\mathbf{k}) u^{+}(\mathbf{k}) \mathbf{h}^{+}(\mathbf{k}) + c^{-}(\mathbf{k}) u^{-}(\mathbf{k}) \mathbf{h}^{-}(\mathbf{k})$$

$$c^{\pm}(\mathbf{k}) = \frac{\varepsilon_E(\mathbf{k}) \pm \varepsilon_H(\mathbf{k})/k}{4E^{\pm}(\mathbf{k})}$$

Solving the full Navier-Stokes equations

Separated injection scales of energy and helicity



Solving the full Navier-Stokes equations

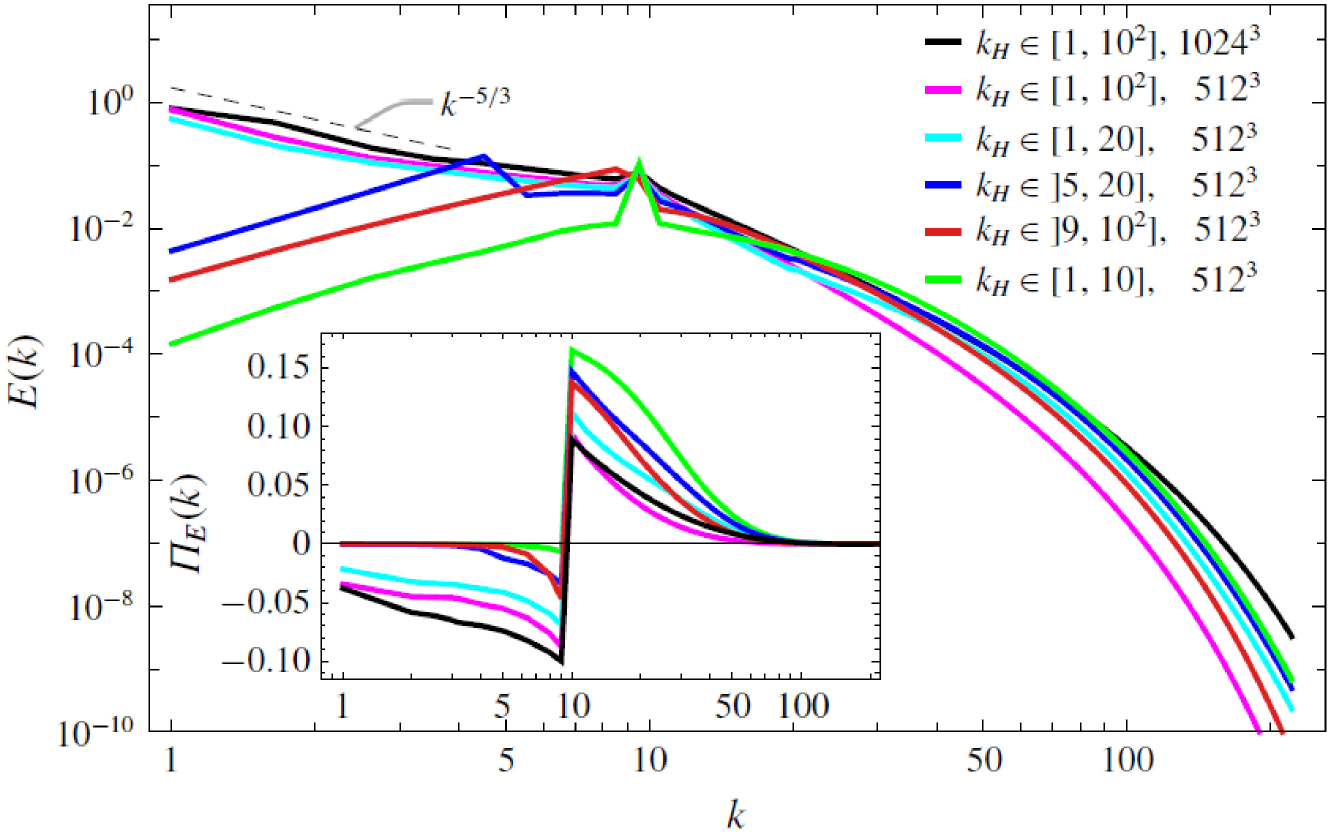


FIGURE 9. Energy spectra and fluxes (inset). The energy injection rate $\varepsilon_E = 0.2$ is applied at $k_E \in]9, 10]$. From bottom to top of the energy spectra, the helicity injection rate $\tilde{\varepsilon}_H = 25$ is applied at $k_H \in [1, 10]$, $]9, 10^2]$, $]5, 20]$, $[1, 20]$, $[1, 10^2]$ and $[1, 10^2]$ again. For the last curve the viscosity is twice as small and the resolution is equal to 1024^3.

Solving the full Navier-Stokes equations

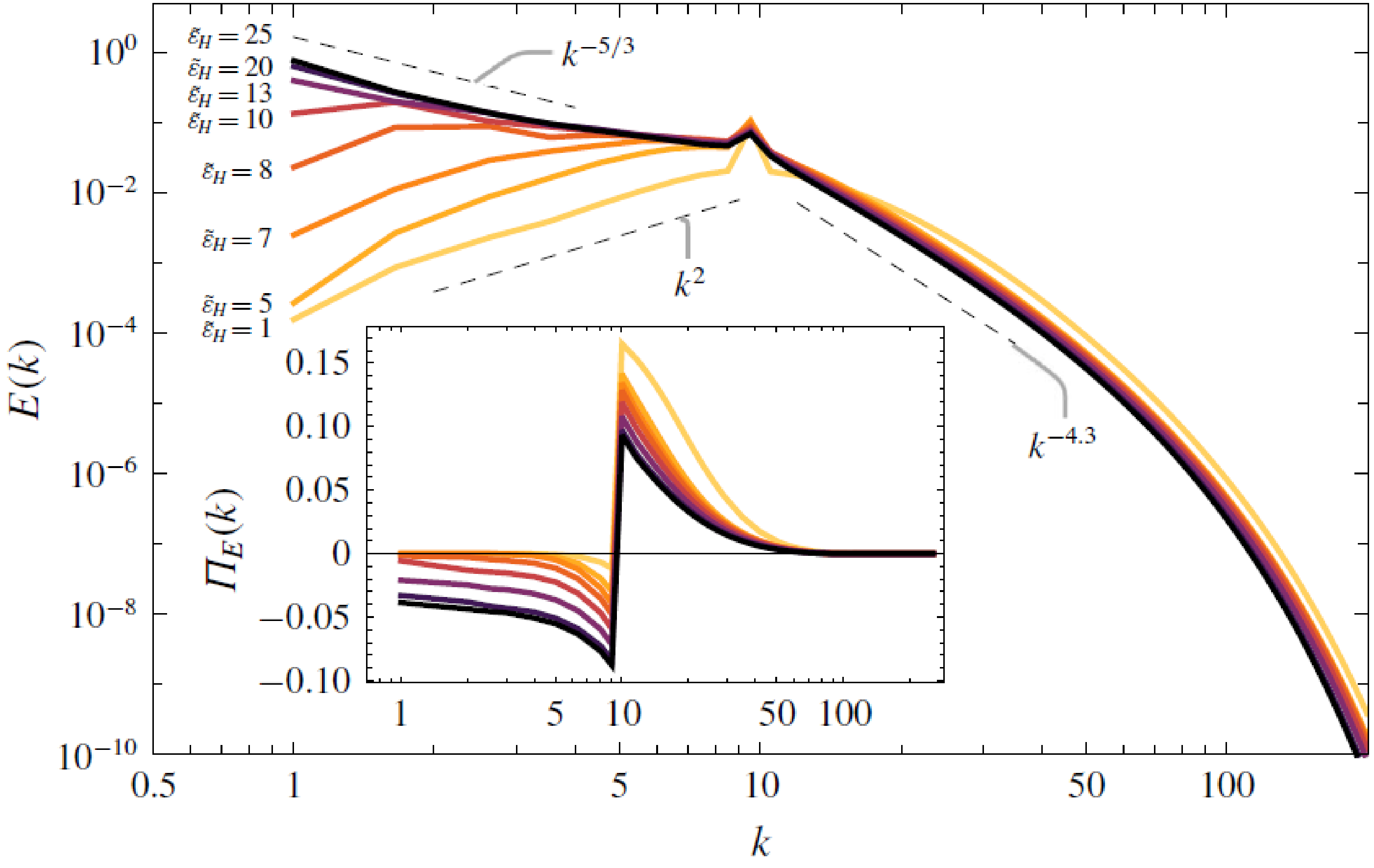


FIGURE 2. Energy spectra and fluxes (inset). The energy injection rate $\varepsilon_E = 0.2$ is applied at $k_E \in]9, 10]$. The helicity injection rate $\tilde{\varepsilon}_H \in \{1; 5; 7; 8; 10; 13; 20; 25\}$ is applied at $k_H \in [1, 10^2]$.

Solving the full Navier-Stokes equations

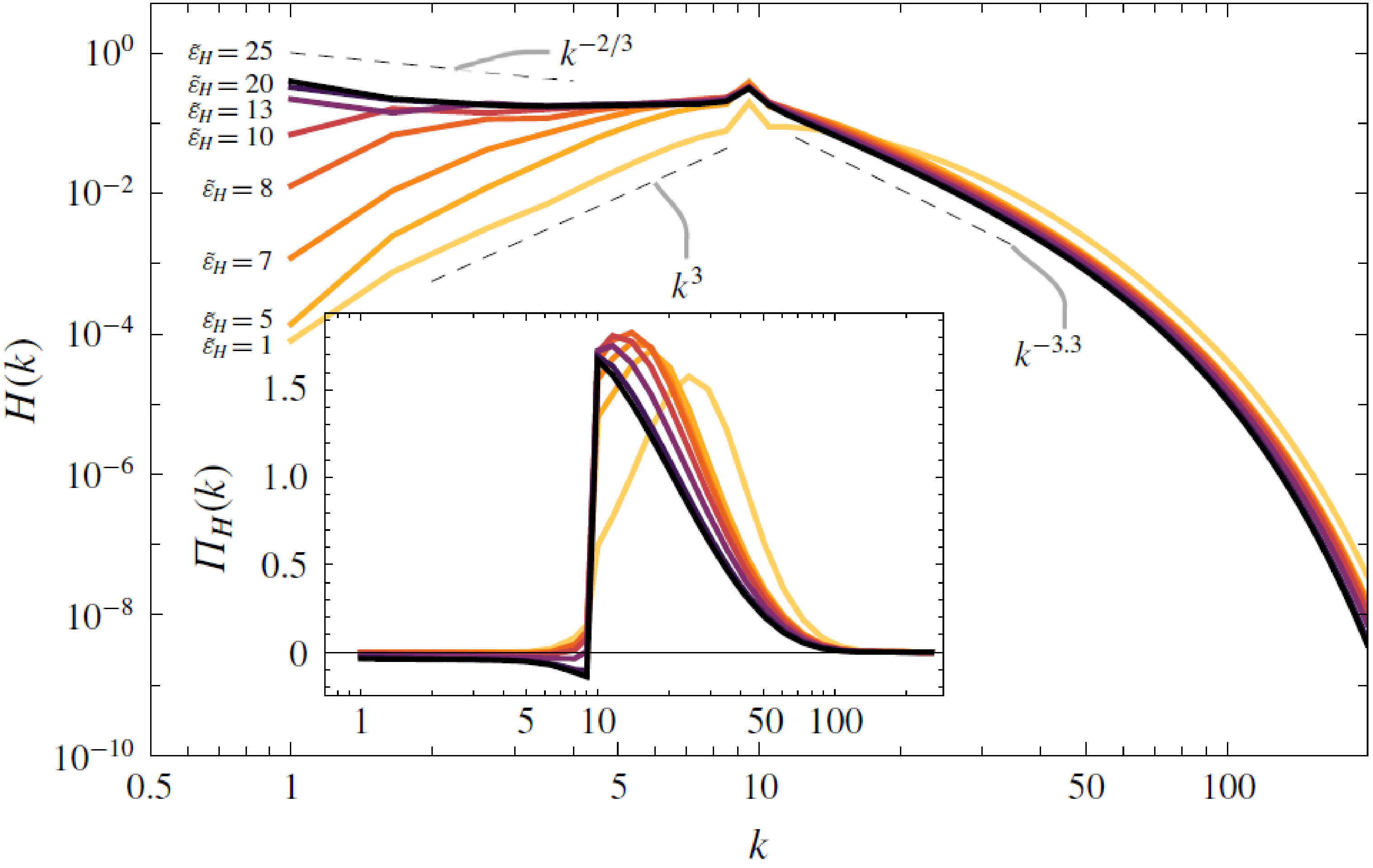


FIGURE 3. Helicity spectra and fluxes (inset) for the same parameters as in figure 2.

Solving the full Navier-Stokes equations

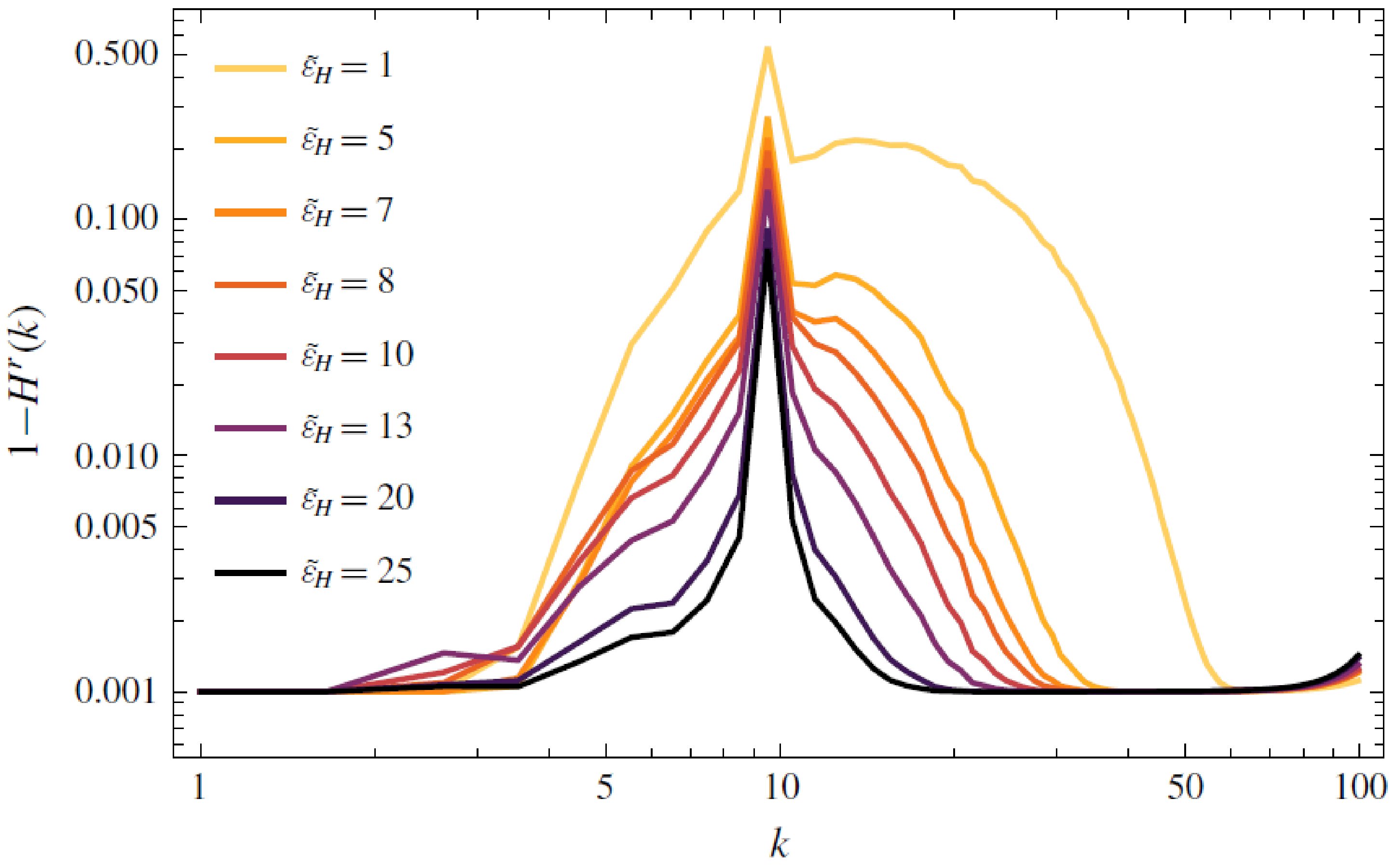
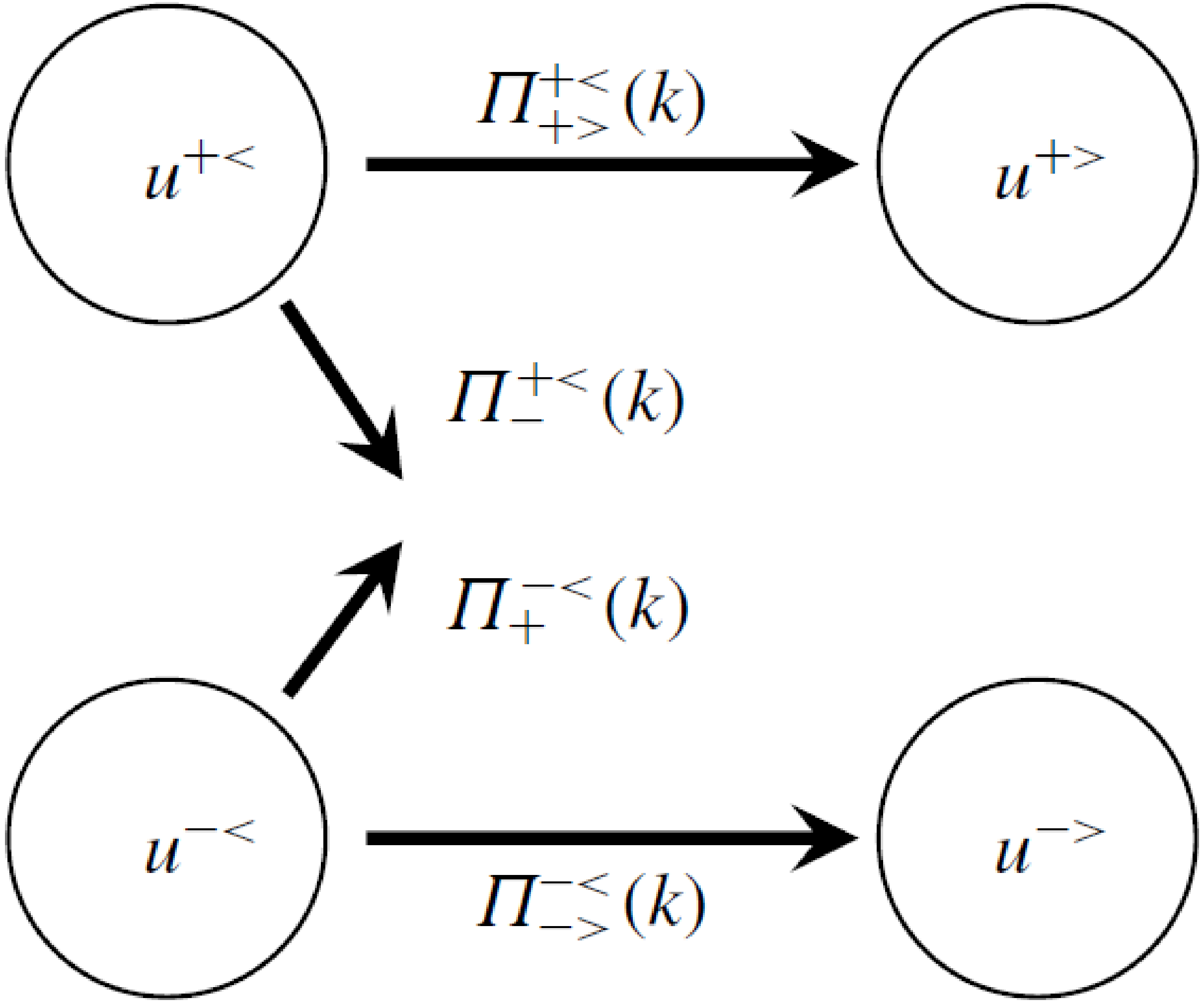
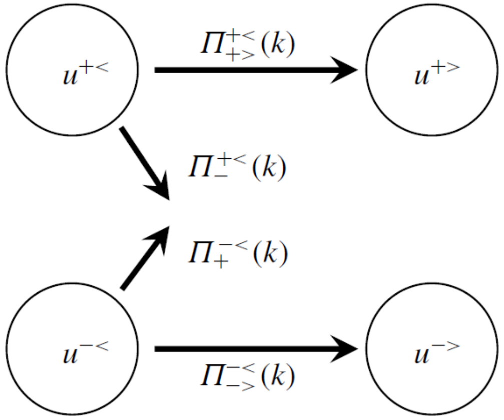


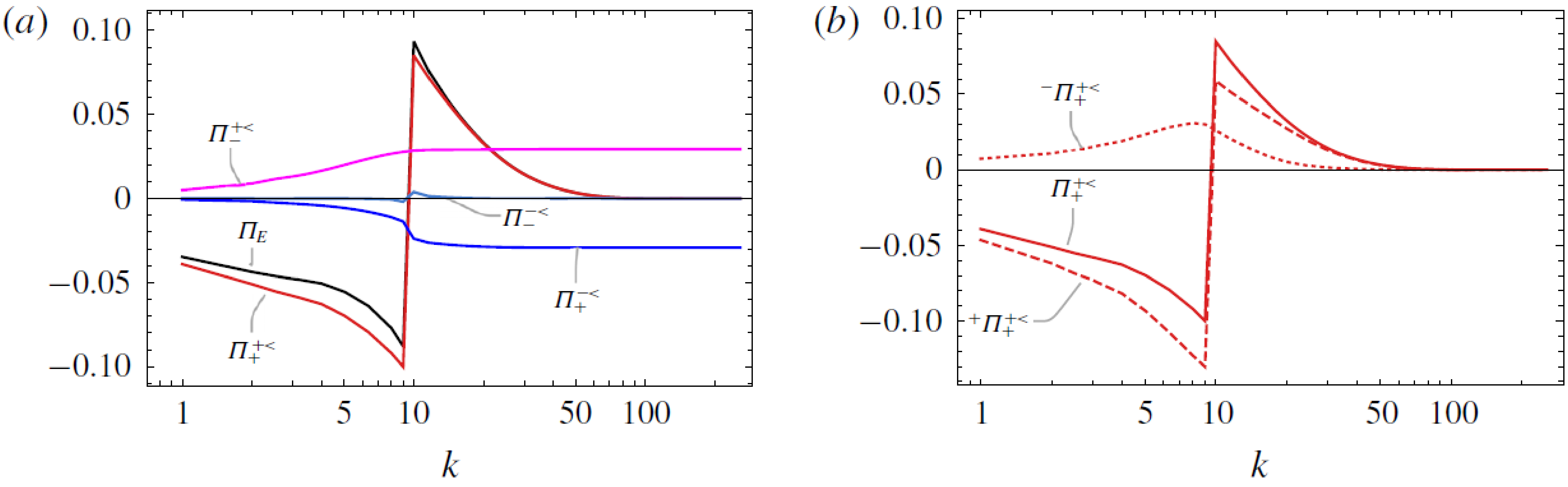
FIGURE 4. Deviation to maximum chirality, $1 - H^r(k)$, for the same parameters as in figure 2.

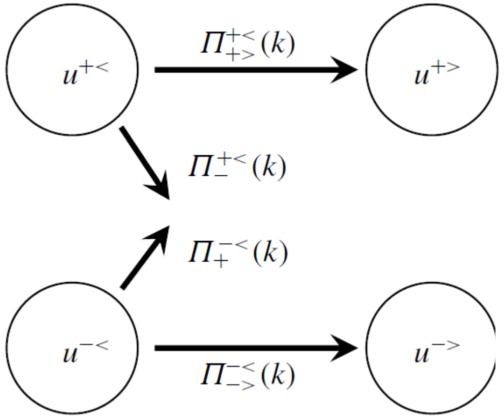
Helical modes decomposition



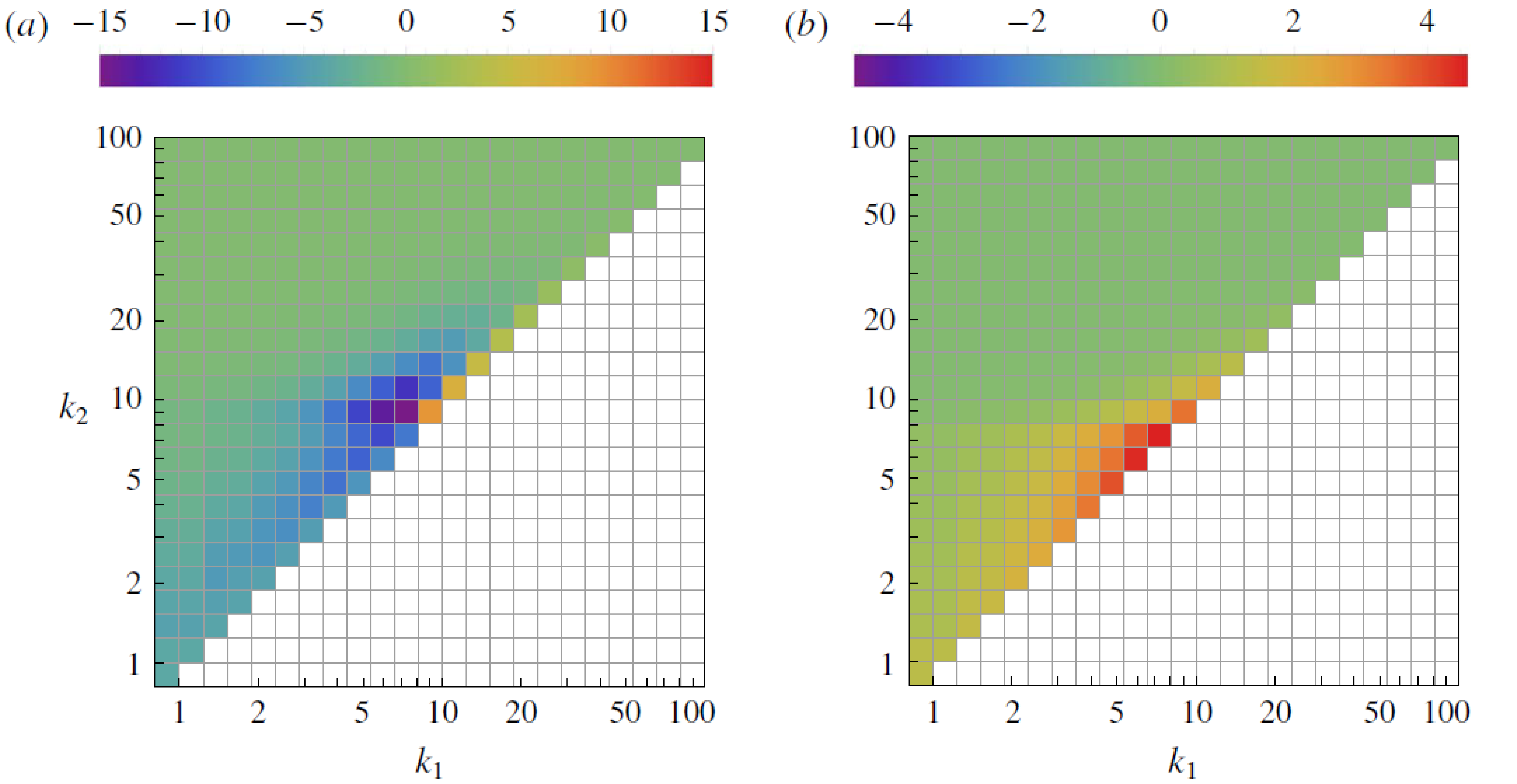
Helical modes decomposition







Helical modes decomposition



Fluxes among homochiral triads

Fluxes among heterochiral triads

Velocity Helicity

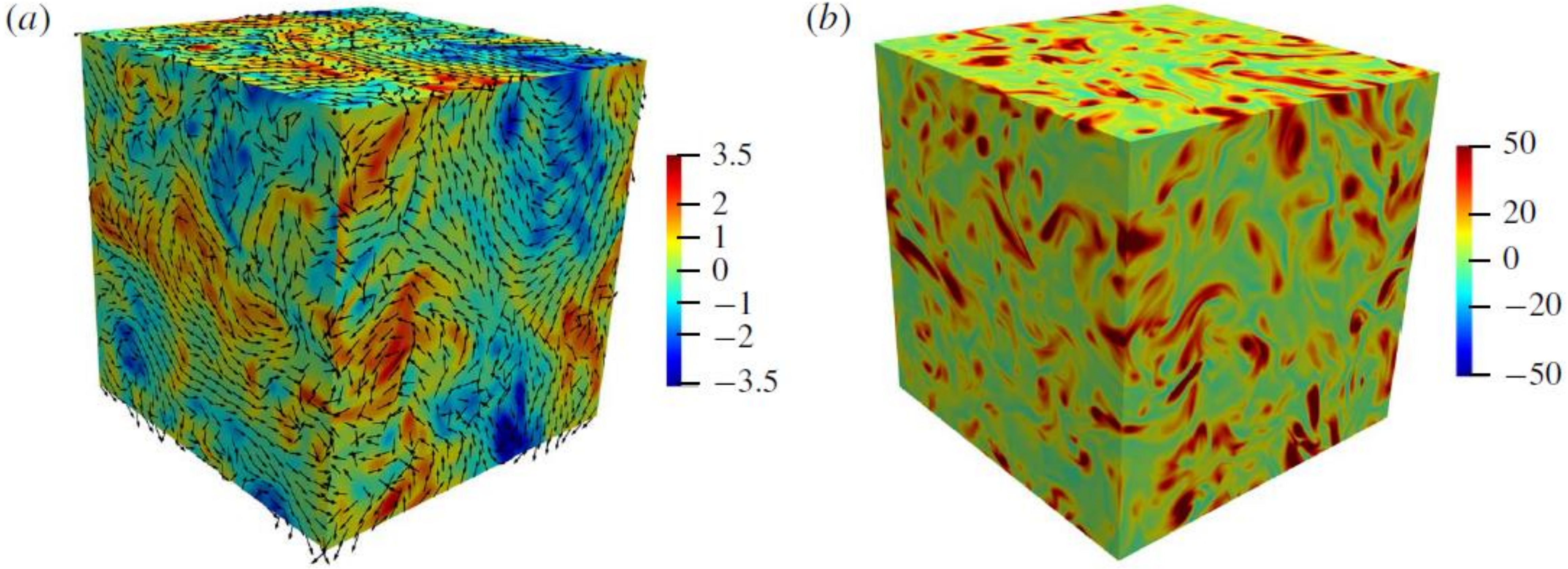


FIGURE 10. Snapshots of velocity (a) and helicity (b) on the three faces of the cubic resolution domain for $\varepsilon_E = 0.2$ and $\tilde{\varepsilon}_H = 25$. In ($a$) the colours represent the isovalues of the velocity component perpendicular to each face, and the arrows the velocity field parallel to each face. In (b) the colours represent the isovalues of helicity.

Conclusions

- Using 3D DNS we find a dual cascade, inverse for energy and direct for helicity.
- A necessary condition is that positive (or negative) helicity is injected in a sufficiently broad range of scales, on both sides of the energy injection range of scales.
- The triads responsible for the inverse cascade of energy are the triads of positive (resp. negative) helical modes, consistent with the homochiral framework.
- The turbulence is fully 3D

Reference: Plunian et al., JFM (2020)