Kinematic dynamo in a tetrahedron of Fourier modes

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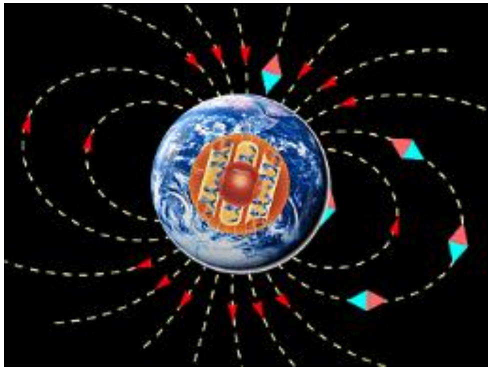
What is the lowest possible dimension of dynamo model?
What kind of helicity do we need for a dynamo?

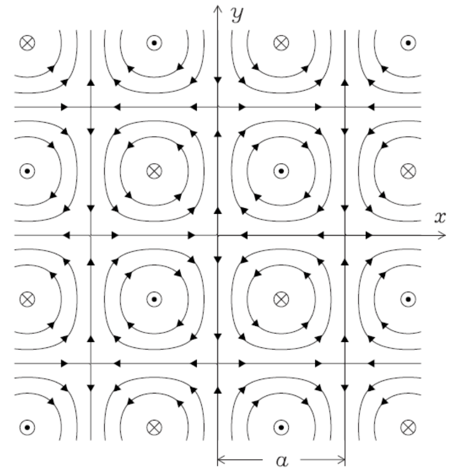
R. Stepanov and F. Plunian, “Kinematic dynamo in a tetrahedron of Fourier modes,” Fluid Dynamics Research, vol. 50, 2018

Dynamo problem and alpha-effect

Geodynamo



Roberts dynamo(1970)

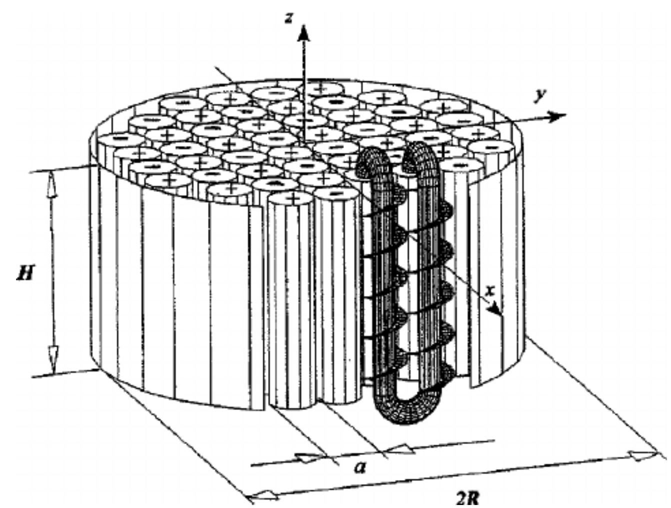


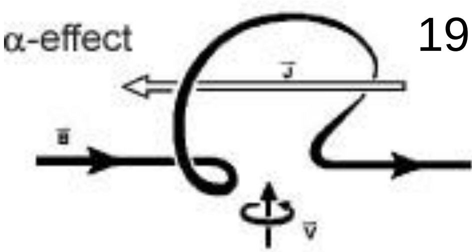
$$\nabla\omega$$

$$B_P \quad \alpha\omega\text{ –dynamo} \quad B_T$$

α– effect

Dynamo experiment in Karlsruhe (1999)





1966

$$\alpha = -\frac{1}{3}\tau_c \overline{\boldsymbol{u}' \cdot \boldsymbol{\omega}'}$$

helicity

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Helicity is Unnecessary for Alpha Effect Dynamos, But it Helps

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We present a two-scale analysis showing that helicity is not required for an alpha effect and associated dynamo instability, and that lack of parity-invariance in the velocity field is sufficient. We give an example of a non-helical velocity field which supports alpha effect dynamo action and demonstrate this effect numerically.

Zero helicity dynamo

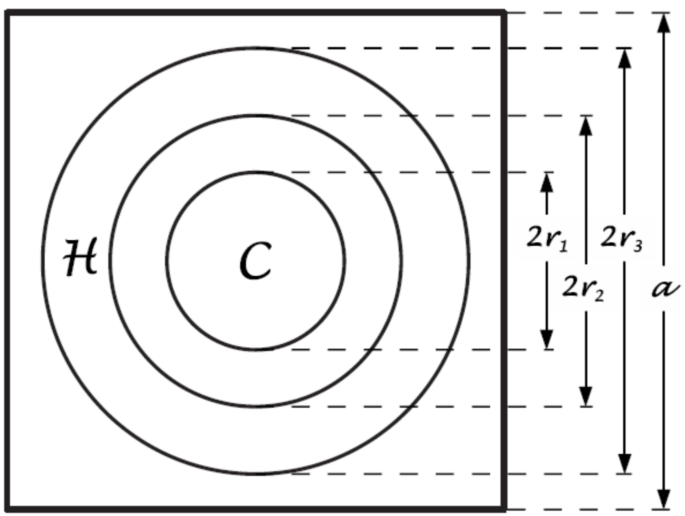
$$\nabla \times \boldsymbol{\psi} = \boldsymbol{u}$$

(Rädler and Brandeburg, 2008)

$$\alpha_{\perp} = \frac{1}{2\eta}\overline{\boldsymbol{u}\cdot\boldsymbol{\psi}}$$

- determines a dynamo threshold

Model flow



$$u_r = 0 \quad \text{everywhere},$$

$$u_\varphi = 0, \quad u_z = u \quad \text{in } r \leqslant r_1,$$

$$u_\varphi = 0, \quad u_z = 0 \quad \text{in } r_1 < r < r_2,$$

$$u_\varphi = \omega r, \quad u_z = \varepsilon\omega a/2 \quad \text{in } r_2 \leqslant r \leqslant r_3,$$

$$\overline{\boldsymbol{u}\cdot\boldsymbol{\omega}} = c\varepsilon V_H^2 \qquad \overline{\boldsymbol{u}\cdot\boldsymbol{\psi}} = (c_1 V_C + \varepsilon c_2 V_H) V_H$$

Mirror symmetry should be broken at an individual scale (Fourier mode) of the flow

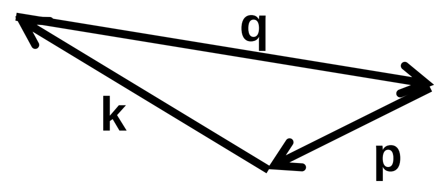
Helical decomposition

Induction equation

$$\left(\partial_t - \eta\nabla^2\right)\mathbf{b}(\mathbf{x}) = \nabla \times \left(\mathbf{u}(\mathbf{x}) \times \mathbf{b}(\mathbf{x})\right)$$

Fourier transform

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \mathbf{b}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{b}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$



Induction equation in terms of Fourier modes

$$\left(\partial_t + \eta k^2\right)\mathbf{b}(\mathbf{k}) = \sum_{\substack{\mathbf{p},\mathbf{q} \\ \mathbf{k}+\mathbf{p}+\mathbf{q}=0}} i\mathbf{k} \times \left(\mathbf{u}^*(\mathbf{p}) \times \mathbf{b}^*(\mathbf{q})\right)$$

Decomposition in helical modes

Waleffe (1992)

$$i\mathbf{k} \times \mathbf{h}^{\pm} = \pm k \mathbf{h}^{\pm}$$

$$\begin{aligned} \mathbf{u}(\mathbf{k}) &= u^+(\mathbf{k})\,\mathbf{h}^+(\mathbf{k}) + u^-(\mathbf{k})\,\mathbf{h}^-(\mathbf{k}), \\ \mathbf{b}(\mathbf{k}) &= b^+(\mathbf{k})\,\mathbf{h}^+(\mathbf{k}) + b^-(\mathbf{k})\,\mathbf{h}^-(\mathbf{k}). \end{aligned}$$

$\Rightarrow$

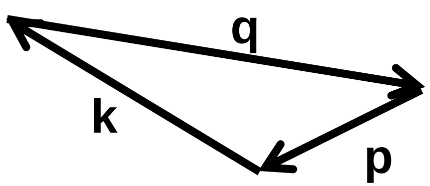
$$\begin{aligned} E(\mathbf{k}) &= |u^+(\mathbf{k})|^2 + |u^-(\mathbf{k})|^2 \\ H(\mathbf{k}) &= k\left(|u^+(\mathbf{k})|^2 - |u^-(\mathbf{k})|^2\right) \end{aligned}$$

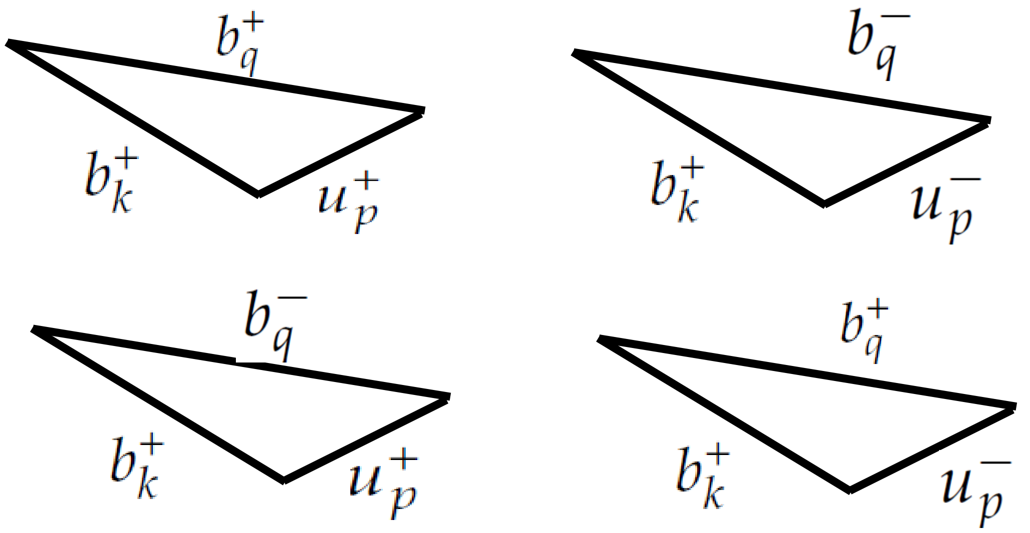
Induction equation in terms of helical modes

$$\left(d_t + \eta k^2\right) b^{s_k}(\mathbf{k}) = -s_k\, k \sum_{\Delta_{\mathbf{kpq}}} \sum_{s_p, s_q = \pm 1} g_{s_k, s_p, s_q}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \left(u^{s_p}(\mathbf{p}) b^{s_q}(\mathbf{q}) - b^{s_p}(\mathbf{p}) u^{s_q}(\mathbf{q})\right)^*$$

$$g_{s_k, s_p, s_q}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \equiv -\frac{1}{\mathbf{h}^{s_k}(\mathbf{k})^* \cdot \mathbf{h}^{s_k}(\mathbf{k})} \left(\mathbf{h}^{s_k}(\mathbf{k})^* \times \mathbf{h}^{s_p}(\mathbf{p})^*\right) \cdot \mathbf{h}^{s_q}(\mathbf{q})^*$$

Helical triads





Variables:	unknown	given
	$b_k^{\pm}, b_p^{\pm}, b_q^{\pm}$	$u_k^{\pm}, u_p^{\pm}, u_q^{\pm}$

Kinematic dynamo in a single triad

Full system wth η=0

$$
\begin{aligned}
d_t b_k^+ &= -k\left[g_{+,+,+}(u_p^+ b_q^+ - b_p^+ u_q^+)^* + g_{+,-,-}(u_p^- b_q^- - b_p^- u_q^-)^* + g_{+,+,-}(u_p^+ b_q^- - b_p^+ u_q^-)^* + g_{+,-,+}(u_p^- b_q^+ - b_p^- u_q^+)^*\right] \\
d_t b_p^+ &= -p\left[g_{+,+,+}(u_q^+ b_k^+ - b_q^+ u_k^+)^* + g_{-,+,-}(u_q^- b_k^- - b_q^- u_k^-)^* + g_{-,+,+}(u_q^+ b_k^- - b_q^+ u_k^-)^* + g_{+,+,-}(u_q^- b_k^+ - b_q^- u_k^+)^*\right] \\
d_t b_q^+ &= -q\left[g_{+,+,+}(u_k^+ b_p^+ - b_k^+ u_p^+)^* + g_{-,-,+}(u_k^- b_p^- - b_k^- u_p^-)^* + g_{+,-,+}(u_k^+ b_p^- - b_k^+ u_p^-)^* + g_{-,+,+}(u_k^- b_p^+ - b_k^- u_p^+)^*\right] \\
d_t b_k^- &= +k\left[g_{-,+,+}(u_p^+ b_q^+ - b_p^+ u_q^+)^* + g_{-,-,-}(u_p^- b_q^- - b_p^- u_q^-)^* + g_{-,+,-}(u_p^+ b_q^- - b_p^+ u_q^-)^* + g_{-,-,+}(u_p^- b_q^+ - b_p^- u_q^+)^*\right] \\
d_t b_p^- &= +p\left[g_{+,-,+}(u_q^+ b_k^+ - b_q^+ u_k^+)^* + g_{-,-,-}(u_q^- b_k^- - b_q^- u_k^-)^* + g_{-,-,+}(u_q^+ b_k^- - b_q^+ u_k^-)^* + g_{+,-,-}(u_q^- b_k^+ - b_q^- u_k^+)^*\right] \\
d_t b_q^- &= +q\left[g_{+,+,-}(u_k^+ b_p^+ - b_k^+ u_p^+)^* + g_{-,-,-}(u_k^- b_p^- - b_k^- u_p^-)^* + g_{+,-,-}(u_k^+ b_p^- - b_k^+ u_p^-)^* + g_{-,+,-}(u_k^- b_p^+ - b_k^- u_p^+)^*\right]
\end{aligned}
$$

$$d_t X = A X^* \qquad X = (b_k^+, b_p^+, b_q^+, b_k^-, b_p^-, b_q^-)$$

$$d_t^2 X = A A^* X$$

Look for an exponential growth rate

$$\gamma = \Re\{a^{1/2}\} = \max_{i=1,6}\left(\Re\{a_i^{1/2}\}\right)$$

Solution

$$a = -\frac{Q_{kpq}^2}{4}\left(\left|\frac{u_k^+ - u_k^-}{k}\right|^2 + \left|\frac{u_p^+ - u_p^-}{p}\right|^2 + \left|\frac{u_q^+ - u_q^-}{q}\right|^2\right)$$

a - real
and a<0
no dynamo

where Q_{kpq} is an area of triad (**k**, **p**, **q**)

Kinematic dynamo in a single helical triad

$u^{s_k}, b^{s_k}, u^{s_p}, b^{s_p}, u^{s_q}, b^{s_q}$

$b^{-s_k} = b^{-s_p} = b^{-s_q} = u^{-s_k} = u^{-s_p} = u^{-s_q} = 0.$

Decimated NS equation

Biferale etal, 2013
Linkmann etal, 2016

System of equations

$$
\begin{aligned}
d_t b^{s_k} &= -s_k k g_{s_k,s_p,s_q}(u^{s_p} b^{s_q} - b^{s_p} u^{s_q})^* \\
d_t b^{s_p} &= -s_p p g_{s_k,s_p,s_q}(u^{s_q} b^{s_k} - b^{s_q} u^{s_k})^* \\
d_t b^{s_q} &= -s_q q g_{s_k,s_p,s_q}(u^{s_k} b^{s_p} - b^{s_k} u^{s_p})^*
\end{aligned}
$$

Solution

$$a = -s_k s_p s_q k p q |g_{s_k,s_p,s_q}|^2 F$$

with $F = s_k \frac{|u^{s_k}|^2}{k} + s_p \frac{|u^{s_p}|^2}{p} + s_q \frac{|u^{s_q}|^2}{q}$

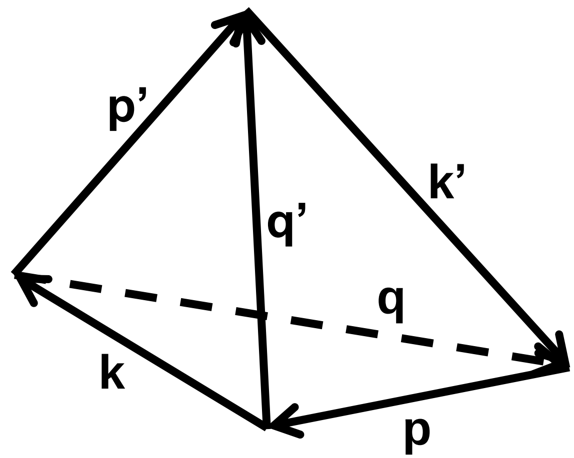
a>0 if

$$\boxed{\begin{aligned} u_k^+ &= u_k^- = 0 \\ s_k &= -s_p = -s_q \end{aligned}}$$

Kinenatic dynamo in tetrahedron

Two scale flow

$u_k^{\pm} \neq 0 \qquad u_p^{\pm} \neq 0$



(**k**, **p**, **q**), (**k**, **p’**, **q’**),
(**k’**, **p’**, **q**), (**k’**, **p**, **q’**)

Solution

$$a = -\frac{Q^2_{kp'q'}}{4}\left|\frac{u_k^+ - u_k^-}{k}\right|^2 - \frac{Q^2_{pq'k'}}{4}\left|\frac{u_p^+ - u_p^-}{p}\right|^2 \pm \frac{Q_{kp'q'}Q_{pq'k'}}{2}\frac{|\sin\psi_{q'}|}{q'}\sqrt{\left(\frac{|u_k^+|^2 - |u_k^-|^2}{k}\right)\left(\frac{|u_p^+|^2 - |u_p^-|^2}{p}\right)}$$

$\psi_{q'}$ - angle between planes of triads (**k**, **p’**, **q’**) and (**k’**, **p**, **q’**)

Necessary condition $\boxed{H(\mathbf{k})H(\mathbf{p}) \neq 0}$

Maximum growth rate

$$\boxed{\begin{array}{l} q' \ll k \text{ and } q' \ll p \\ \psi_{q'} = \pi/2 \qquad \mathbf{q}' \perp (\mathbf{k}, \mathbf{p}, \mathbf{q}) \end{array}}$$

R. Stepanov and F. Plunian, Fluid dynamics Research, 2018

Roberts dynamo

Roberts flow (Type I)

$$\boldsymbol{u} = \{\sin(y), \sin(x), \cos(x) - \cos(y)\}$$

$$H = 2(1 - \cos(x)\cos(y))$$

$$\boldsymbol{k} = \{1, 0, 0\} \qquad u^+(\boldsymbol{k}) = \frac{1}{2} \qquad u^-(\boldsymbol{k}) = 0$$

$$\boldsymbol{p} = \{0, 1, 0\} \qquad u^+(\boldsymbol{p}) = -\frac{1}{2} \qquad u^-(\boldsymbol{p}) = 0$$

Modified flow (Type II)

$$\boldsymbol{u} = \{\sin(y), \sin(x), \cos(x) + \cos(y)\}$$

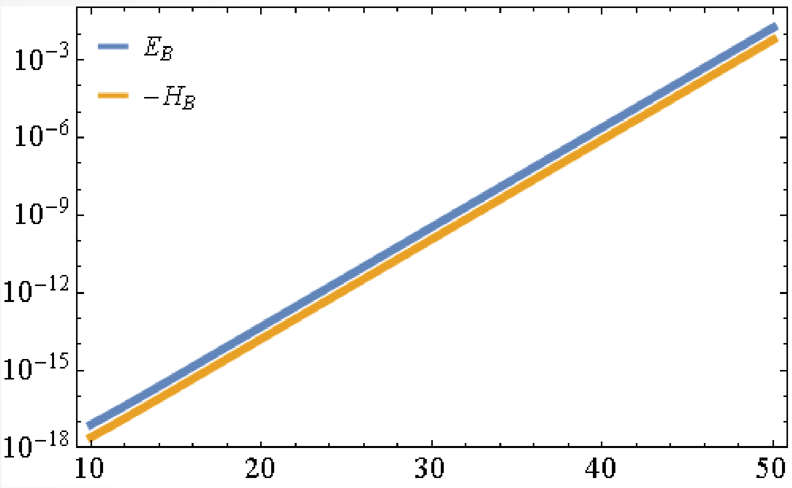
$$H = 0$$

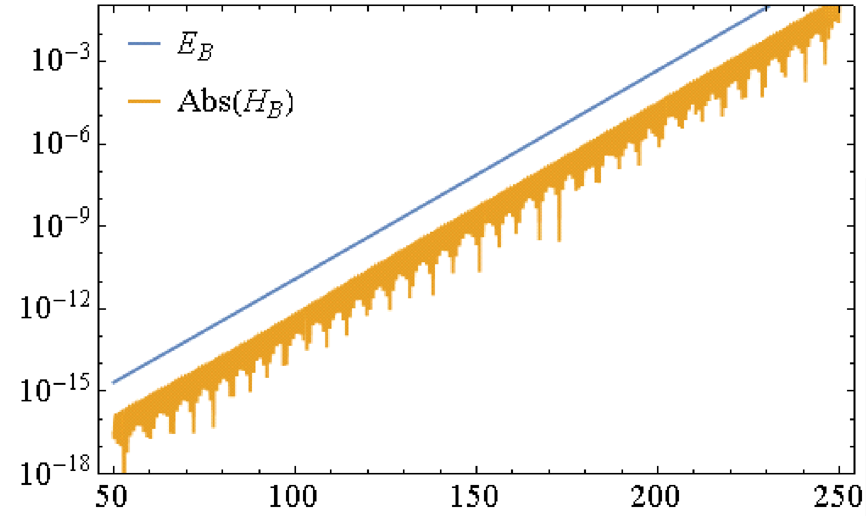
$$\boldsymbol{k} = \{1, 0, 0\} \qquad u^+(\boldsymbol{k}) = \frac{1}{2} \qquad u^-(\boldsymbol{k}) = 0$$

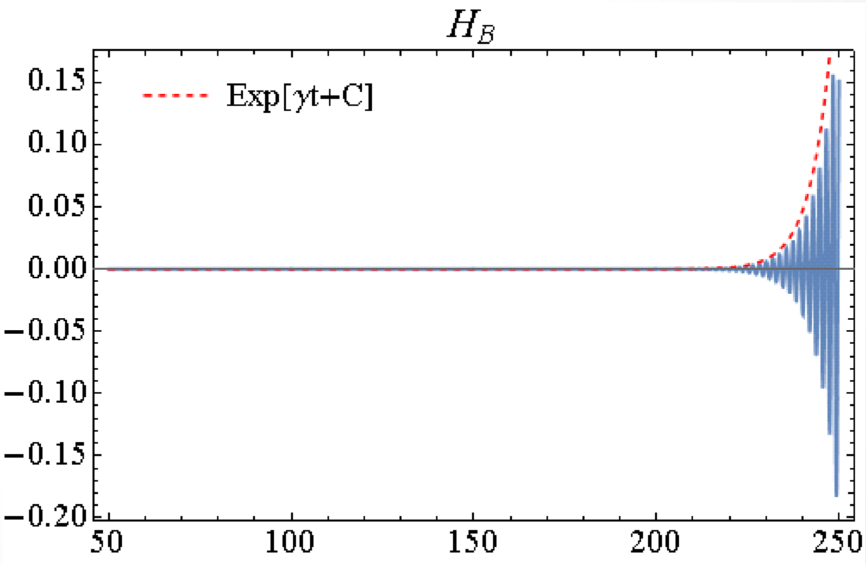
$$\boldsymbol{p} = \{0, 1, 0\} \qquad u^+(\boldsymbol{p}) = 0 \qquad u^-(\boldsymbol{p}) = \frac{1}{2}$$

Time dependences of integral characteristics

Roberts 1	Roberts 2



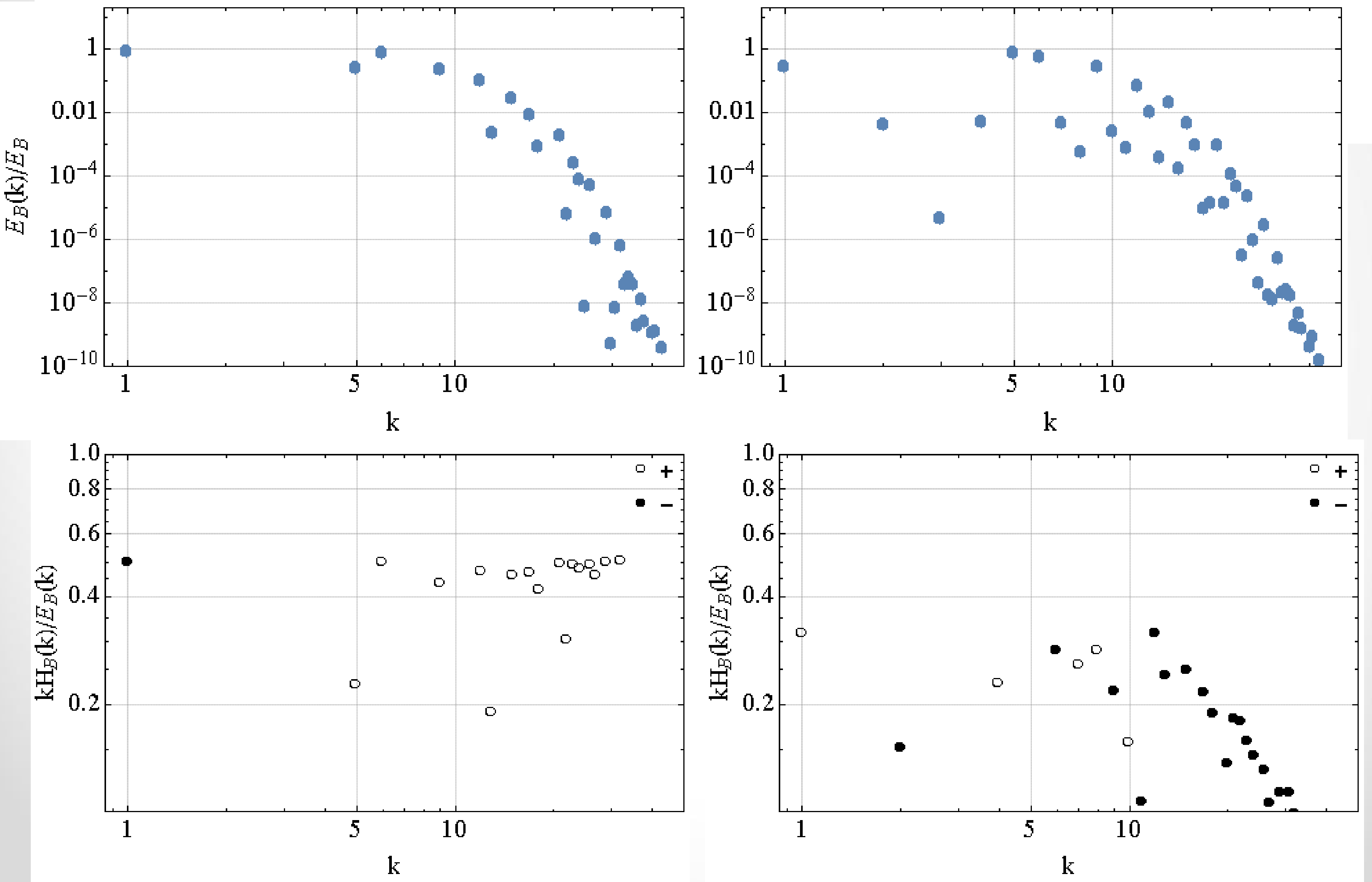




Velocity is given at k={4,0,0} and p={0,4,0}

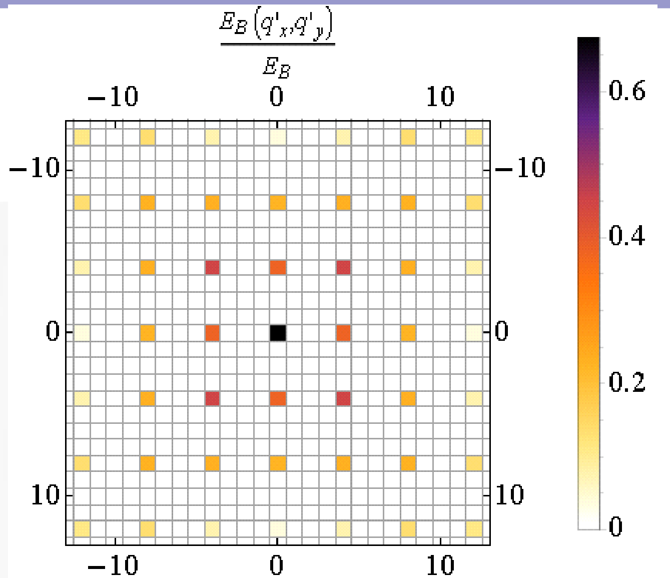
Energy and helicity density spectra

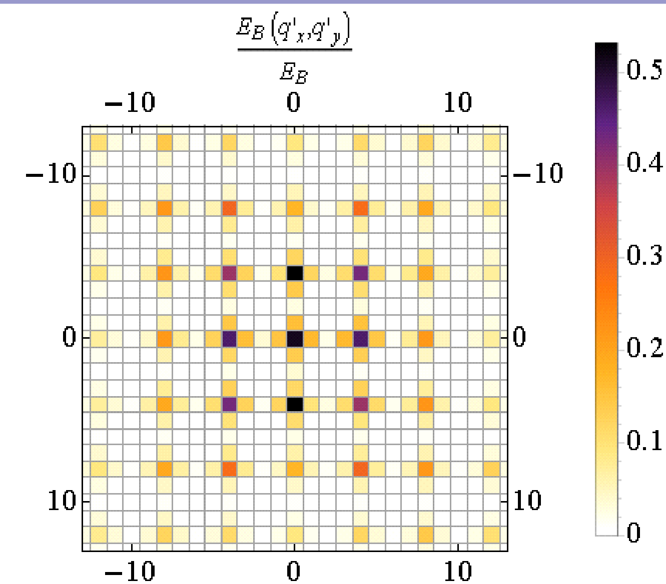
Roberts 1	Roberts 2

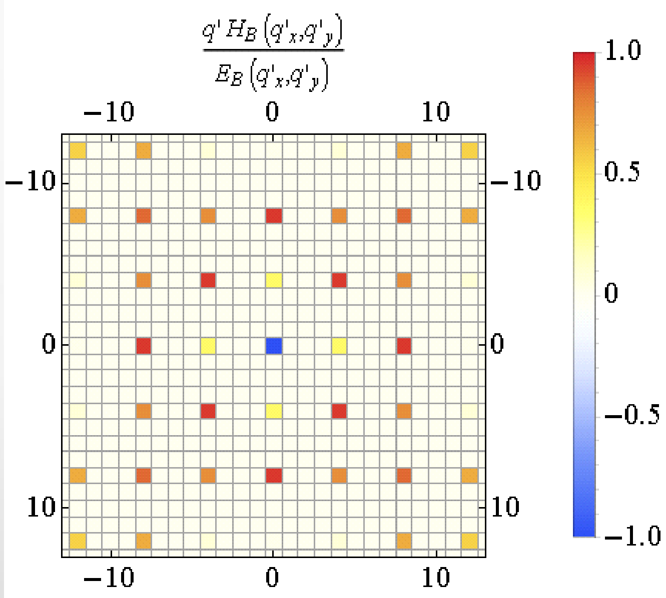


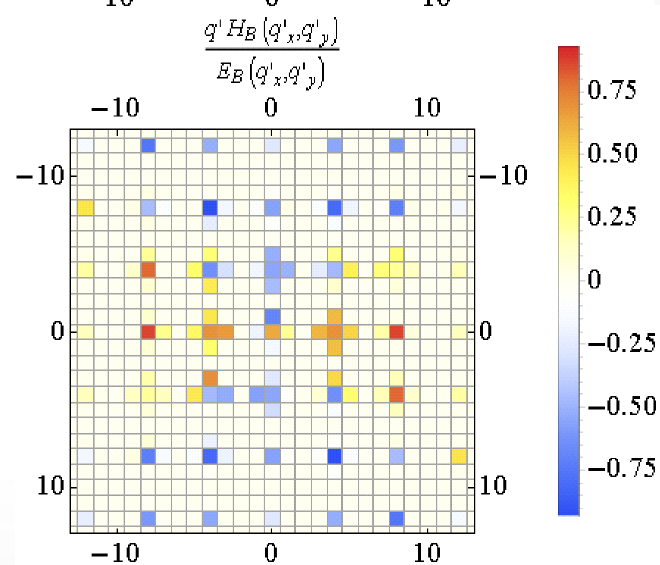
Mode energy and helicity distributions in (kx, ky) plane and kz=1

Roberts 1	Roberts 2



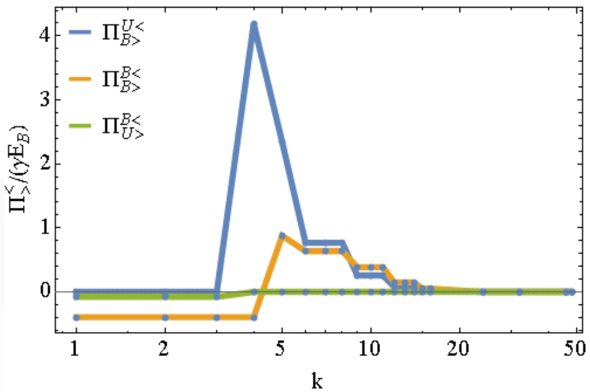


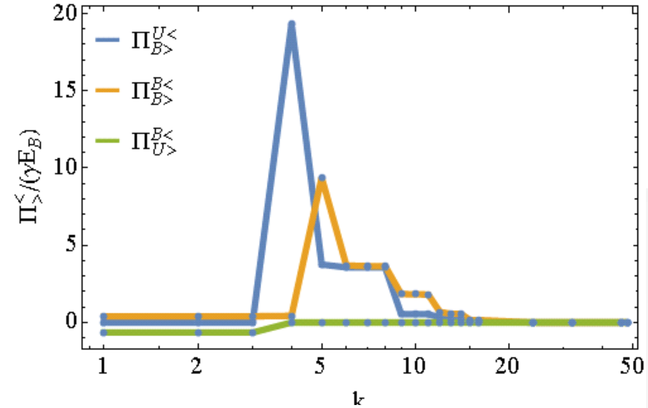


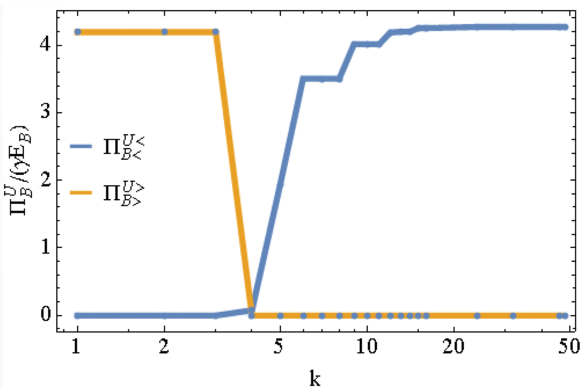


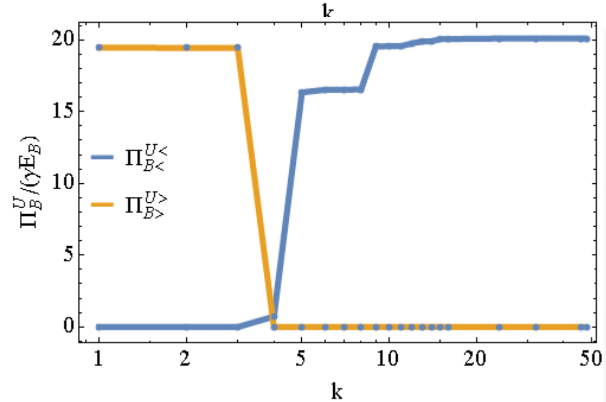
Energy and helicity fluxes

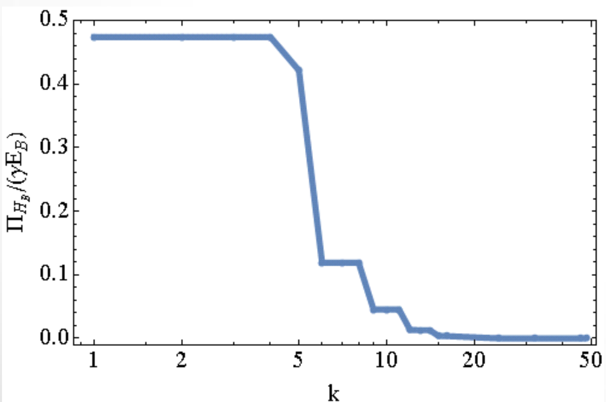
Roberts 1	Roberts 2

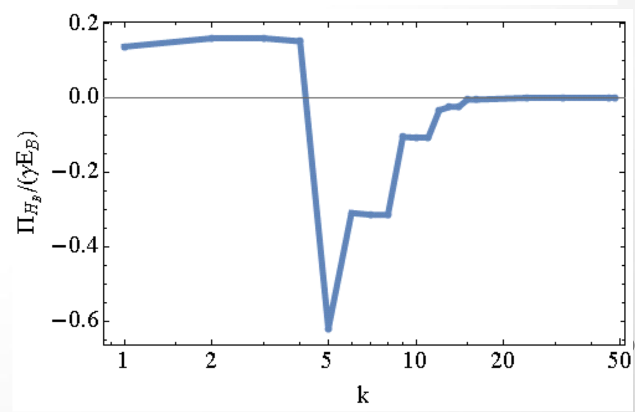






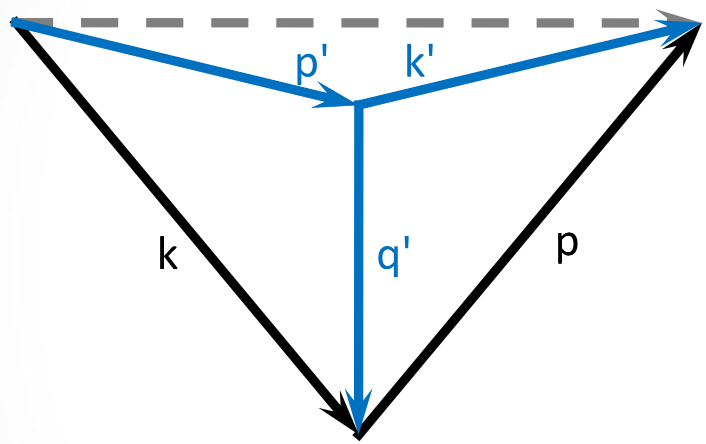






Mode-to-mode transfer

Roberts 1	Roberts 2



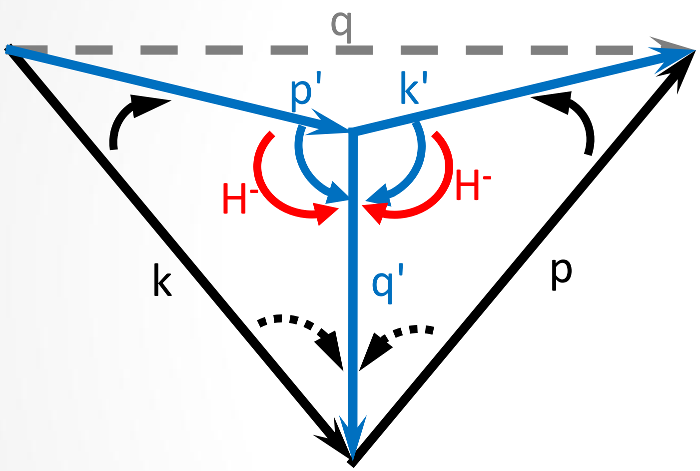
Two acting triads:
{k p’ q’} and {p k’ q}

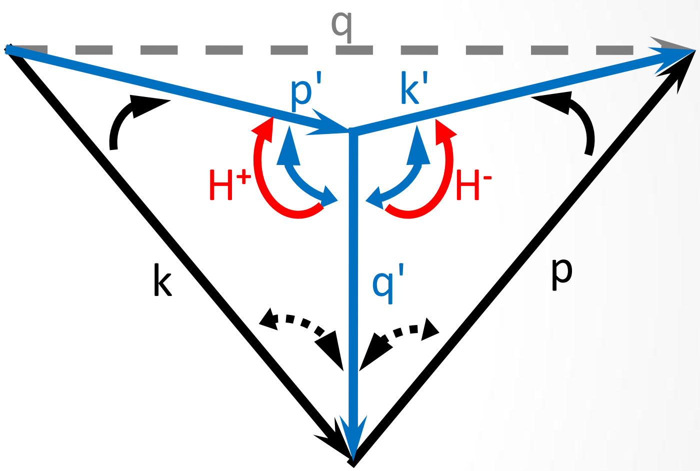
$T^E(U_{-p} \xrightarrow{B_{q'}} B_{k'})$	$T^E(U_k \xrightarrow{B_{q'}} B_{-p'})$	$T^E(U_k \xrightarrow{B_{-p'}} B_{q'})$	$T^E(U_{-p} \xrightarrow{B_{k'}} B_{q'})$
0.875557	0.875557	0.0347682	0.0347682
$T^E(B_{q'} \xrightarrow{U_{-p}} B_{k'})$	$T^E(B_{q'} \xrightarrow{U_k} B_{-p'})$	$T^E(B_{-p'} \xrightarrow{U_k} B_{q'})$	$T^E(B_{k'} \xrightarrow{U_{-p}} B_{q'})$
−0.175429	−0.175429	0.175429	0.175429
$T^H(B_{q'} \xrightarrow{U_{-p}} B_{k'})$	$T^H(B_{q'} \xrightarrow{U_k} B_{-p'})$	$T^H(B_{-p'} \xrightarrow{U_k} B_{q'})$	$T^H(B_{k'} \xrightarrow{U_{-p}} B_{q'})$
0.420394	0.420394	−0.420394	−0.420394

$T^E(U_{-p} \xrightarrow{B_{q'}} B_{k'})$	$T^E(U_k \xrightarrow{B_{q'}} B_{-p'})$	$T^E(U_k \xrightarrow{B_{-p'}} B_{q'})$	$T^E(U_{-p} \xrightarrow{B_{k'}} B_{q'})$
1.14806	1.13987	0.0715908	0.0715743
$T^E(B_{q'} \xrightarrow{U_{-p}} B_{k'})$	$T^E(B_{q'} \xrightarrow{U_k} B_{-p'})$	$T^E(B_{-p'} \xrightarrow{U_k} B_{q'})$	$T^E(B_{k'} \xrightarrow{U_{-p}} B_{q'})$
0.0450311	0.0447819	−0.0447819	−0.0450311
$T^H(B_{q'} \xrightarrow{U_{-p}} B_{k'})$	$T^H(B_{q'} \xrightarrow{U_k} B_{-p'})$	$T^H(B_{-p'} \xrightarrow{U_k} B_{q'})$	$T^H(B_{k'} \xrightarrow{U_{-p}} B_{q'})$
−0.538242	0.534141	−0.534141	0.538242

Scheme of mode interactions

Roberts 1	Roberts 2





U-to-B energy transfer

B-to-B energy transfer

Magnetic helicity transfer

Conclusions

- Helical decomposition is powerful tool to deal with the problems related to helicity

- We show that the simplest modes configuration leading to an unstable solution has the form of a tetrahedron

- We pick up main features of Roberts-like dynamos from analytical solution.